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Abstract

We examine a two-bidder auction setting in which the distributions for the bidders' valuations are asymmetric over a support consisting of three elements. For the first price auction we derive the unique Bayes Nash Equilibrium in closed form, which allows to obtain more precise results with respect to the classical results in the literature on how asymmetries affect equilibrium bidding. Then we compare the revenue in the first price auction with the revenue in the second price auction. The latter is often superior to the former and we determine precisely, given a distribution for the value of the weak bidder, when a distribution for the value of the strong bidder exists such that the first price auction is superior to the second price auction. For two particular asymmetries, shift and stretch, we show that in our setting the results are quite different from the results which are well-known in the literature.

1 Introduction

This paper is about an auction setting in which bidders have asymmetrically distributed values, but for which it is possible to characterize in closed form the unique Bayes Nash Equilibrium for the first price auction. This allows to derive quite accurate results on the effects of asymmetries on equilibrium bidding, and on the revenue comparison between the first price auction and the second price auction.

In the standard auction setting, bidders have private values which are ex ante i.i.d. random variables; this delivers many significant results for the standard setting. Conversely, the important and realistic extension in which bidders have asymmetrically distributed values is more difficult to deal with for a variety of auctions, for instance for the first price auction (FPA), because asymmetric distributions often prevent the existence of a closed form for the equilibrium bidding functions[1] – one exception is the second price auction (SPA), in which bidding the own valuation is a weakly dominant strategy for each bidder. This makes it difficult, in an asymmetric environment, to compare the revenues from different auction formats, or to perform comparative statics analysis about the effect of a change in the distributions of the valuations.

In this paper we examine a setting with two bidders in which the valuation of each bidder has the same support $\{v_L, v_M, v_H\}$, with $v_H - v_M = v_M - v_L > 0$, but the probability distribution for v_1, the value of bidder 1, may be different from the probability distribution for v_2, the value of bidder 2.[2] We use λ_i and μ_i, respectively, to denote $\Pr\{v_i = v_L\}$ and $\Pr\{v_i = v_M\}$, respectively, hence $\Pr\{v_i = v_H\} = 1 - \lambda_i - \mu_i$, for $i = 1, 2$. The only restriction we impose on these probabilities is the innocuous inequality

$$\lambda_1 + \mu_1 \leq \lambda_2 + \mu_2 \tag{1}$$

that is $\Pr\{v_1 = v_H\} \geq \Pr\{v_2 = v_H\}$.

We determine in closed form the unique Bayes Nash Equilibrium for the FPA, which involves mixed strategies for both bidders. Our equilibrium characterization allows to identify precisely the effects of asymmetric distributions on equilibrium bidding with respect to a symmetric environment.[3] Specifically, well known results in the literature are that the ex ante weak bidder bids more aggressively than the ex ante strong bidder, but the overall bid distribution of the strong bidder is stronger than the bid distribution of the weak bidder. In particular, first order stochastic dominance between the value distributions suffices for the latter result, and conditional stochastic dominance suffices for the former result.[4] In our setting, given (1), if there is a weak bidder then it is bidder 2 and we show that the overall bid distribution of bidder 1 is stronger than the bid distribution of bidder 2 if and only if (1) holds with strict inequality, a condition less restrictive than first order stochastic dominance.[5] Likewise, we identify a necessary and sufficient condition for bidder 2 to bid more aggressively than bidder 1 which is less restrictive than conditional stochastic dominance.

Then we move to compare the FPA and the SPA in terms of revenue. In this regard, it is useful to notice that the supports of the bids submitted by type 1_H and type 2_H share the same maximum bid, which implies that these types have the same utility, and this typically has the consequence that type 1_M (or type 2_M, but not both) puts a probability mass on the bid v_L. This "mass" feature of the equilibrium in the FPA increases the winning probability and the utility for type 1_M or for type 2_M above the winning probability and the utility under the SPA. In fact, when $\lambda_1 + \mu_1 + \mu_2 < \lambda_2$ we find that type 1_M bids v_L with probability 1, and also type 1_H puts a probability mass on v_L. This occurs because type 1_H earns utility at least $\lambda_2(v_H - v_L)$ if he bids v_L (or just above v_L), which implies that his equilibrium utility, and in particular his utility from the common maximum bid, is no less than $\lambda_2(v_H - v_L)$. Therefore also the utility of type 2_H from the common maximum bid is no less than $\lambda_2(v_H - v_L)$, and if λ_2 is large relative to λ_1, μ_1, μ_2, then $\lambda_2(v_H - v_L)$ turns out to be greater than 2_H's utility from the lowest bid in 2_H's support. This makes 2_H prefer to bid the common maximum bid as a pure strategy, which is not consistent with equilibrium. Equilibrium requires that the utility from the lowest bid in 2_H's support is increased, which occurs when type 1_H bids v_L with a

[1] Plum (1992), Cheng (2006), Kaplan and Zamir (2012) derive equilibrium for the FPA in closed form for specific settings.

[2] Maskin and Riley (1983), Maskin and Riley (1985), Cheng (2011), Doni and Menicucci (2013) examine settings with discretely distributed values, but restrict to cases in which the value of each bidder has a binary support.

[3] The availability of a closed form for the equilibrium strategies allows also to examine a bidder's incentives to invest ex ante in order to improve the value distribution, an issue we briefly discuss in the conclusions, for a procurement setting.

[4] See Lebrun (1998), Maskin and Riley (2000a), Li and Riley (2007), Kirkegaard (2009).

[5] The two distributions are equal if (1) holds with equality.

positive probability such that type 2_H is indifferent between the lowest bid in his support and the common maximum bid. In this case the equilibrium strategies do not depend on λ_1, μ_1.

Since the SPA allocates the object efficiently but the FPA does not, a sufficient condition for R^S, the expected revenue under the SPA, to be higher than R^F, the expected revenue under the FPA, is that the bidders' rents in the FPA are greater than in the SPA. We prove that this often occurs because of the mass feature of the equilibrium for the FPA.[6] More in detail, we show that some probability distributions for v_2 are such that $R^S \geq R^F$ *for each* distribution for v_1: this is the case when $\lambda_2 + \mu_2 \leq \frac{1}{2}$ and/or λ_2 is quite different from μ_2 (see Figure 2 in Subsection 4.2.1). If instead $\lambda_2 + \mu_2 > \frac{1}{2}$ and λ_2 is not too different from μ_2, then there exists at least one distribution for v_1 such that $R^F > R^S$, and it is a distribution which induces the strongest bidding in the FPA by both bidders.

When $\lambda_2 + \mu_2 > \frac{1}{2}$ and λ_2 is large given $\lambda_2 \leq \mu_2$, if $R^F > R^S$ holds for some (λ_1, μ_1) then it holds if $\lambda_1 = 0$, $\mu_1 = 0$, that is if $\Pr\{v_1 = v_H\} = 1$. This makes it intuitive that $R^F > R^S$ requires $\lambda_2 + \mu_2$ not too small, as a small $\lambda_2 + \mu_2$ makes $\Pr\{v_2 = v_H\}$ large, which jointly with $\Pr\{v_1 = v_H\} = 1$ makes likely the state of the world $(v_1, v_2) = (v_H, v_H)$ in which the revenue in the SPA is v_H, the highest possible – this gives the SPA a significant advantage. It is also intuitive why a large λ_2, given $\lambda_2 \leq \mu_2$, makes it more likely that $R^F > R^S$ holds: a large λ_2 increases the probability that $(v_1, v_2) = (v_H, v_L)$ is the state of the world, in which the revenue in the SPA is v_L (the lowest possible) and moreover we can show that when $\lambda_1 = 0$ and $\mu_1 = 0$, bidder 1 bids no less than v_M; thus in each state of the world, included $(v_1, v_2) = (v_H, v_L)$, the revenue from the FPA is no less than v_M. Hence an increase in λ_2 makes more likely a state in which the FPA yields at least $v_M - v_L$ more than the SPA. Moreover, an increase in λ_2 lowers the probability $1 - \lambda_2 - \mu_2$ that the state of the world is $(v_1, v_2) = (v_H, v_H)$, in which the revenue in the SPA is v_H.[7]

When $\lambda_2 + \mu_2 > \frac{1}{2}$ and μ_2 is large given $\lambda_2 > \mu_2$, if $R^F > R^S$ holds for some (λ_1, μ_1) then it holds if $\lambda_1 = \lambda_2 - \mu_2$, $\mu_1 = 0$, but not necessarily if $\lambda_1 = 0$, $\mu_1 = 0$. The reason is that starting from $\lambda_1 = \lambda_2 - \mu_2$, $\mu_1 = 0$, a reduction of λ_1 to 0 would not change bidding in the FPA since we mentioned above that such bidding does not depend on λ_1, μ_1 when $\lambda_1 + \mu_1 + \mu_2 \leq \lambda_2$. But the reduction in λ_1 improves bidder 1's bid distribution in the SPA, hence increases R^S. Given $\lambda_1 = \lambda_2 - \mu_2$, $\mu_1 = 0$, when μ_2 is large given $\lambda_2 > \mu_2$, we have that λ_2 is close to μ_2, a situation analogous to the one described in the above paragraph, with $R^F > R^S$. But a reduction of μ_2 such that μ_2 becomes close to 0 weakens significantly the bidding in the FPA as the overall bid distribution of bidder 1 puts a probability mass of $\lambda_2 - \mu_2$ on the bid v_L which increases as μ_2 decreases. Actually, given $\lambda_1 = \lambda_2 - \mu_2$, the reduction of μ_2 weakens the bidding also in the SPA, but such effect is dominated by the former effect and as a result $R^S > R^F$ when μ_2 is close to zero.

Maskin and Riley (1985) prove that $R^S > R^F$ always holds in a setting in which each bidder's value has a (same) binary support. Conversely, in our setting with ternary support it possible that R^F is greater than R^S. We explain that this occurs because starting from a symmetric setting, with $R^F = R^S$, a suitable improvement in a bidder's value distribution increases R^F and R^S, which in some cases results in $R^F > R^S$. But when the support is binary, any improvement in the value distribution of a bidder has the effect of increasing R^S, while R^F does not change as neither bidder changes his bid distribution in FPA.

Finally, we examine the shift model and the stretch model introduced in Maskin and Riley (2000a), for which Kirkegaard (2012) establishes that the FPA generates a higher revenue than the SPA, under slightly more general assumptions than Maskin and Riley (2000a). For each of these models we determine precisely when $R^F > R^S$ in our setting and obtain significantly different results with respect to Kirkegaard (2012).

[6] Conversely, the literature has identified several settings in which the opposite result, $R^F > R^S$, holds: see for instance Maskin and Riley (2000a), Li and Riley (2007), Kirkegaard (2012), Kirkegaard (2014), Kirkegaard (2021).

[7] The reverse of this argument suggests why $R^S > R^F$ when λ_2 is about zero.

Precisely, in the shift model $R^S > R^F$ holds unless the weak distribution (from which the strong distribution is obtained through a rightward shift) is relatively strong. In the stretch model, when the weak distribution (from which the strong distribution is obtained through a stretch) is very weak, we find that any significant stretch leads to $R^S > R^F$.

The rest of the paper is organized as follows: Section 2 introduces the auction environment. Section 3 is about equilibrium bidding in the FPA and the effect of asymmetry on equilibrium bidding. Section 4 compares the FPA and the SPA in terms of revenue. Section 5 concludes. The appendix provides the proofs for some of our results. The missing proofs are available in Ceesay, Doni, Menicucci (2024).

2 Model

A (female) seller owns an object to which she attaches no value and faces two (male) bidders interested in buying the object. Bidder 1 (bidder 2) privately observes his own monetary value v_1 (v_2) for the object, which is equal either to v_L, or to v_M, or to v_H, with $v_L \geq 0$ and $v_M = v_L + \Delta$, $v_H = v_M + \Delta$ for a positive Δ. Bidder 2 and the seller view v_1 as a realization of a random variable for which the probabilities of v_L, v_M, v_H are denoted with λ_1, μ_1, η_1:

$$\lambda_1 = \Pr\{v_1 = v_L\} > 0, \qquad \mu_1 = \Pr\{v_1 = v_M\} > 0, \qquad \eta_1 = \Pr\{v_1 = v_H\} > 0$$

Likewise, bidder 1 and the seller view v_2 as a realization of a random variable – which is stochastically independent of v_1 – for which λ_2, μ_2, η_2 denote the probabilities of v_L, v_M, v_H:[8]

$$\lambda_2 = \Pr\{v_2 = v_L\} > 0, \qquad \mu_2 = \Pr\{v_2 = v_M\} > 0, \qquad \eta_2 = \Pr\{v_2 = v_H\} > 0$$

Although the two random variables have the same support $\{v_L, v_M, v_H\}$, they are asymmetrically distributed unless $(\lambda_1, \mu_1, \eta_1) = (\lambda_2, \mu_2, \eta_2)$. The expected utility of each bidder is given by his value times his probability to win the object, minus his expected payment. The seller is risk neutral.

3 Equilibrium bidding

3.1 Equilibrium bidding in the FPA

When the seller offers the object through a FPA, each bidder simultaneously submits a sealed bid, the highest bidder wins and pays his bid to the seller. For some tie-breaking rules, no pure-strategy equilibrium exists in this game, but Proposition 2 in Maskin and Riley (2000b) establishes that an equilibrium, possibly in mixed strategies, exists under the "Vickrey tie-breaking rule", according to which in the in the FPA each bidder i is required to submit both an "ordinary" bid $b_i \geq 0$ and a "tie-breaker" bid $c_i \geq 0$.[9] The tie-breaking rule (see Maskin and Riley (2000b) for a complete description) specifies that c_1, c_2 matter only when $b_1 = b_2$, and implies that for each bidder i it is weakly dominant to submit a tie-breaking bid c_i equal to $v_i - b_i$; hence, in describing a strategy of bidder i in the following we implicitly assume that to each b_i is associated $c_i = v_i - b_i$. As a result, when $b_1 = b_2$ the bidder with the highest value wins and pays to the seller the value of the other bidder.

[8] Although we require here $\lambda_i > 0, \mu_i > 0, \eta_i > 0$ for $i = 1, 2$, in the following we consider sometimes the case in which some of the above probabilities are zero. In such case the equilibrium can be obtained by applying a limit argument to the equilibrium obtained when $\lambda_i > 0, \mu_i > 0, \eta_i > 0$ for $i = 1, 2$.

[9] A very similar idea appears in Lebrun (2002), in the auction denoted with $F\bar{P}A$.

Proposition 1 in Subsection 3.1.2 identifies, for each parameter values, an equilibrium for the FPA with the Vickrey tie-breaking rule. Next subsection introduces some notation and basic equilibrium properties.

3.1.1 Some equilibrium properties

We use i_j to denote type j of bidder i, for $j = L, M, H$ and $i = 1, 2$, and since we need a notation for mixed strategies, we let G_{ij} denote the c.d.f. of the bid submitted by type i_j. We denote with u_{ij}^F the equilibrium expected utility of type i_j.

By relying on arguments in Maskin and Riley (1985) and Riley (1989), we deduce that each equilibrium satisfies the following properties.

1. Type 1_L and type 2_L both bid v_L with probability 1 (a pure strategy), that is G_{1L} and G_{2L} both put probability 1 on the bid v_L. Hence $u_{1L}^F = 0$, $u_{2L}^F = 0$.

2. For each G_{ij}, the set of possible realizations of G_{ij} is an interval and we denote with $\underline{b}_{ij}$ and $\bar{b}_{ij}$ the infimum and the supremum of such interval, respectively. Moreover, $\underline{b}_{1M} = v_L$, $\underline{b}_{1H} = \bar{b}_{1M}$, and $\underline{b}_{2M} = v_L$, $\underline{b}_{2H} = \bar{b}_{2M}$. That is, there are no gaps in the distribution of bids by each bidder.

3. Each G_{ij} is a continuous function, except possibly at $b = v_L$ when $\underline{b}_{ij} = v_L$. That is, no bidder puts a probability mass on a single bid, except possibly for the bid v_L.

4. A mixed strategy G_{ij} is a best response for type i_j if and only if each bid in the set of the possible realizations of G_{ij} maximizes the expected utility of type i_j; hence type i_j needs to be indifferent among all such bids.

5. The highest bid for type 1_H, $\bar{b}_{1H}$, is equal to the highest bid for type 2_H, $\bar{b}_{2H}$, that is $\bar{b}_{1H} = \bar{b}_{2H}$. In the following we use $\bar{b}_H$ to denote the common value of $\bar{b}_{1H}, \bar{b}_{2H}$ and notice that $u_{1H}^F = u_{2H}^F = v_H - \bar{b}_H$ because of remark 4 and since either bidder wins with probability 1 by bidding $\bar{b}_H$.

From remark 3 we deduce that if a bidder bids $b > v_L$, then with probability 1 no tie occurs and we can evaluate the expected utility of a bidder – say bidder 1 – from a bid $b > v_L$ as follows. We define G_2 as the c.d.f. of the bids submitted by bidder 2, that is $G_2(b) = \lambda_2 G_{2L}(b) + \mu_2 G_{2M}(b) + \eta_2 G_{2H}(b)$. Then if type j of bidder 1 submits a bid $b > v_L$, his expected utility is $(v_j - b)G_2(b)$ because $G_2(b)$ is his probability to win. Likewise, $G_1(b) = \lambda_1 G_{1L}(b) + \mu_1 G_{1M}(b) + \eta_1 G_{1H}(b)$ is the c.d.f. of the bids submitted by bidder 1, and for type 2_j the expected utility from bidding $b > v_L$ is $(v_j - b)G_1(b)$.

From remark 4 we deduce that for type 1_M the equality $(v_M - b)G_2(b) = u_{1M}^F$ holds for each $b \in (v_L, \bar{b}_{1M}]$, that is type 1_M is indifferent among all bids in the interval $(v_L, \bar{b}_{1M}]$. Moreover, u_{1M}^F coincides with $\lim_{b \downarrow v_L}(v_M - b)G_2(b)$, that is with $G_2(v_L)\Delta$. Since type 2_L bids v_L, we know that $G_2(v_L) \geq \lambda_2$ but we cannot rule out that also type 2_M bids v_L with positive probability. Hence it is possible that $G_2(v_L) > \lambda_2$, and we use ρ_2 to denote $G_2(v_L)$; likewise, we set $\rho_1 = G_1(v_L)$. Therefore

$$u_{1M}^F = \rho_2 \Delta \quad \text{with} \quad \rho_2 \geq \lambda_2 \qquad \text{and} \qquad u_{2M}^F = \rho_1 \Delta \quad \text{with} \quad \rho_1 \geq \lambda_1 \tag{2}$$

Notice that it cannot happen that both type 1_M and type 2_M bid v_L with positive probability, as in such a case either type would have incentive to increase the own bid slightly above v_L to increase, by a discrete amount, his probability to win while increasing his expected payment just infinitesimally. Hence $\rho_1 \geq \lambda_1$, $\rho_2 \geq \lambda_2$ but at least one of these weak inequalities holds as equality.

Applying remark 4 to types $1_M, 1_H, 2_M, 2_H$ we obtain

$$(v_M - b)G_2(b) = \rho_2\Delta \quad \text{for each} \quad b \in (v_L, \bar{b}_{1M}] \tag{3}$$
$$(v_H - b)G_2(b) = v_H - \bar{b}_H \quad \text{for each} \quad b \in [\bar{b}_{1M}, \bar{b}_H] \tag{4}$$
$$(v_M - b)G_1(b) = \rho_1\Delta \quad \text{for each} \quad b \in (v_L, \bar{b}_{2M}] \tag{5}$$
$$(v_H - b)G_1(b) = v_H - \bar{b}_H \quad \text{for each} \quad b \in [\bar{b}_{2M}, \bar{b}_H] \tag{6}$$

Example: The case of binary support As an example, we illustrate here the role played by ρ_1, ρ_2 when for each bidder there are just two possible values, v_L and v_H (with $v_H - v_L = 2\Delta$), with probabilities λ_1 and $1-\lambda_1$ for bidder 1, λ_2 and $1-\lambda_2$ for bidder 2 (that is, $\mu_1 = \mu_2 = 0$) and $\lambda_1 \leq \lambda_2$.[10] If $\lambda_1 = \lambda_2$, then we obtain $\bar{b}_H = v_H - 2\lambda_2\Delta$ and $G_1(b) = G_2(b) = \frac{v_H - \bar{b}_H}{v_H - b}$ for each $b \in [v_L, \bar{b}_H]$, with $\rho_1 = \lambda_1 = \rho_2 = \lambda_2$. When $\lambda_1 < \lambda_2$, we find that $\bar{b}_H$ is unchanged because type 1_H's equilibrium utility is still $2\lambda_2\Delta$, his utility from bidding v_L. Hence also type 2_H's equilibrium utility is $2\lambda_2\Delta$, and type 2_H needs to earn the same utility $2\lambda_2\Delta$ from any bid in $(v_L, \bar{b}_H]$. However, if type 1_H puts no probability mass on v_L, then 2_H's utility from a bid $b \in (v_L, \bar{b}_H)$ tends to $2\lambda_1\Delta$ as b tends to v_L, rather than to $2\lambda_2\Delta$. Hence in equilibrium ρ_1 needs to be equal to λ_2, that is $\rho_1 > \lambda_1$, which requires that type 1_H bids v_L with probability $\frac{\lambda_2 - \lambda_1}{1-\lambda_1} > 0$. As a result, when λ_1 smaller than λ_2 the c.d.f.s G_1, G_2 and the revenue do not depend on λ_1.

We prove below that in our setting with three possible types for each bidder, sometimes it is bidder 1 who bids v_L with probability ρ_1 greater than λ_1, sometimes it is bidder 2 who bids v_L with probability $\rho_2 > \lambda_2$.

3.1.2 The equilibrium strategies

The c.d.f.s G_1, G_2 are straightforward to derive from (3)-(6) as a function of $\bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H, \rho_1, \rho_2$.[11] In order to determine $\bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H, \rho_1, \rho_2$, without loss of generality we assume that (1) holds. This means that bidder 1 is ex ante weakly stronger than bidder 2 in the sense that $1 - \lambda_1 - \mu_1 = \Pr\{v_1 = v_H\}$ is no less than $1 - \lambda_2 - \mu_2 = \Pr\{v_2 = v_H\}$. But notice that (1) does not imply that the distribution of v_1 first order stochastic dominates the distribution of v_2: that requires $\lambda_1 \leq \lambda_2$ in addition to (1) (with at least one strict inequality). Next lemma shows that (1) implies $\bar{b}_{1M} \leq \bar{b}_{2M}$, thus if bidder 1 turns out to have value v_M, then he is less aggressive in terms of the set of possible bid realizations then type 2_M, and a similar result holds if bidder 1 turns out to have value v_H, relative to type 2_H.[12]

Lemma 1 If $\lambda_1 + \mu_1 < \lambda_2 + \mu_2$, then $v_L \leq \bar{b}_{1M} < \bar{b}_{2M}$; if $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$, then $v_L < \bar{b}_{1M} = \bar{b}_{2M}$. Moreover, $G_1(\bar{b}_{2M}) = \lambda_2 + \mu_2$.

Lemma 1 follows from the property that types 1_H and 2_H earn the same expected utility, that is $u_{1H}^F = u_{2H}^F$ (see remark 5), and a contradiction is obtained if we suppose that $\bar{b}_{2M} \leq \bar{b}_{1M}$.[13] Then (i) by bidding $\bar{b}_{1M}$, which belongs to the interval $[\bar{b}_{2M}, \bar{b}_H]$ in (6), type 2_H beats types 1_L and 1_M by definition of $\bar{b}_{1M}$, hence wins with probability $\lambda_1 + \mu_1$; (ii) by bidding $\bar{b}_{1M}$, which belongs to the interval $[\bar{b}_{1M}, \bar{b}_H]$ in (4), type 1_H wins with probability at least $\lambda_2 + \mu_2$. Since we are considering the same bid $\bar{b}_{1M}$, the equality $u_{1H}^F = u_{2H}^F$ implies that the probabilities to win with the bid $\bar{b}_{1M}$ are equal for types 1_H and 2_H, which

[10] This setting has already been examined in Maskin and Riley (1985), which establish the result about the revenue comparison mentioned just before Subsection 4.1. Here we use this setting as a sort of benchmark.

[11] Once G_1 is determined, it is possible to derive G_{1M}, G_{1H} using $G_1(b) = \lambda_1 + \mu_1 G_{1M}(b) + \eta_1 G_{1H}(b)$ for $b > v_L$ and the equalities $G_{1M}(b) = 1$ for $b \geq \bar{b}_{1M}$, $G_{1H}(b) = 0$ for $b < \bar{b}_{1M}$. Similarly, from G_2 it is possible to derive G_{2M}, G_{2H}.

[12] Proposition 2(i-ii) below compares bidding by the types mentioned above in terms of first order stochastic dominance.

[13] In order to simplify the argument, we assume here $\bar{b}_{2M} > v_L$. The proof of Lemma 1 covers also the case in which $\bar{b}_{2M} = v_L$.

requires $\bar{b}_{1M} = \bar{b}_{2M}$ and that (1) holds with equality – indeed, in such case Lemma 1 establishes $\bar{b}_{1M} = \bar{b}_{2M}$. But if (1) holds strictly, then a contradiction emerges; hence $\bar{b}_{1M} < \bar{b}_{2M}$ must hold. Moreover, the bid $\bar{b}_{2M}$ yields type 1_H a probability to win equal to $\lambda_2 + \mu_2$, hence arguing as above reveals that $\lambda_2 + \mu_2$ is the probability to win of type 2_H by bidding $\bar{b}_{2M}$, that is $G_1(\bar{b}_{2M}) = \lambda_2 + \mu_2$.

In order to fix the ideas, we begin with the case in which $v_L < \bar{b}_{1M}$. Then by Lemma 1 the bids $\bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H$ satisfy

$$v_L < \bar{b}_{1M} \le \bar{b}_{2M} < \bar{b}_H \tag{7}$$

From (5) evaluated at $b = \bar{b}_{1M}$ and at $b = \bar{b}_{2M}$ we derive $\bar{b}_{1M}, \bar{b}_{2M}$ as a function of ρ_1:[14]

$$\bar{b}_{1M} = v_M - \frac{\rho_1}{\lambda_1 + \mu_1}\Delta, \qquad \bar{b}_{2M} = v_M - \frac{\rho_1}{\lambda_2 + \mu_2}\Delta, \qquad \bar{b}_H = v_H - \rho_2(1 + \frac{\lambda_1 + \mu_1}{\rho_1})\Delta \tag{8}$$

Then (3) at $b = \bar{b}_{1M}$ yields $G_2(\bar{b}_{1M}) = \frac{\rho_2}{\rho_1}(\lambda_1 + \mu_1)$, which can be used to evaluate (4) at $\bar{b}_{1M}$ and to derive $\bar{b}_H$ in (8). Therefore the equilibrium is fully determined if ρ_1, ρ_2 are identified. This is achieved by evaluating (6) at $\bar{b}_{2M}$, which reduces to

$$F(\rho_1, \rho_2) = 0, \qquad \text{with} \qquad F(\rho_1, \rho_2) = \rho_2(1 + \frac{\lambda_1 + \mu_1}{\rho_1}) - \rho_1 - \lambda_2 - \mu_2 \tag{9}$$

In particular (omitting the common factor Δ), $\rho_2(1 + \frac{\lambda_1+\mu_1}{\rho_1})$ is type 2_H's utility from bidding $\bar{b}_H$ and $\rho_1 + \lambda_2 + \mu_2$ is 2_H's utility from $\bar{b}_{2M}$.[15]

The equation $F(\rho_1, \rho_2) = 0$ determines ρ_1, ρ_2 uniquely as F is strictly decreasing with respect to ρ_1, strictly increasing with respect to ρ_2, and $\rho_1 \ge \lambda_1$, $\rho_2 \ge \lambda_2$ with at least one equality. On this basis Proposition 1 below identifies, for each given parameter values, a unique equilibrium, which is one of the following three strategy profiles:

$$P_{2M}: \left\{ \begin{array}{c} \text{the distributions of bids are given by } G_1, G_2 \text{ satisfying (3)-(6), with } \bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H \\ \text{in (8) and } \rho_1 = \lambda_1,\ \rho_2 = \lambda_1 \frac{\lambda_1+\lambda_2+\mu_2}{2\lambda_1+\mu_1} \text{ is the unique solution to } F(\lambda_1, \rho_2) = 0 \end{array} \right. \tag{10}$$

$$P_{1M}: \left\{ \begin{array}{c} \text{the distributions of bids are given by } G_1, G_2 \text{ satisfying (3)-(6), with } \bar{b}_{1M}, \bar{b}_{2M}, \bar{b}_H \text{ in (8) and} \\ \rho_2 = \lambda_2,\ \rho_1 = \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2(\lambda_1 + \mu_1)} - \frac{1}{2}\mu_2 \text{ is the unique solution to } F(\rho_1, \lambda_2) = 0 \text{ in } [\lambda_1, \lambda_1 + \mu_1) \end{array} \right. \tag{11}$$

$$P_{1MH}: \left\{ \begin{array}{c} \text{type } 1_M \text{ bids } v_L \text{ (that is, } \bar{b}_{1M} = v_L\text{); the distributions of bids are given by } G_1, G_2 \\ \text{satisfying (4)-(6), with } \bar{b}_{2M} \text{ in (8), } \bar{b}_H = v_H - 2\lambda_2\Delta \text{ and } \rho_1 = \lambda_2 - \mu_2,\ \rho_2 = \lambda_2 \end{array} \right. \tag{12}$$

In each of these profiles, types 1_L and 2_L both bid v_L. The profiles mainly differ because of the additional bidder types who bid v_L with positive probability: in P_{2M} it is only type 2_M; in P_{1M} it is only type 1_M; in P_{1MH}, both type 1_M (with probability 1) and type 1_H bid v_L with positive probability.

Proposition 1 Suppose that (1) is satisfied. Then the unique equilibrium in the FPA is P_{2M} if $F(\lambda_1, \lambda_2) < 0$, that is if

$$\lambda_2(\lambda_1 + \mu_1) < \lambda_1(\lambda_1 + \mu_2) \tag{13}$$

The unique equilibrium is P_{1M} if $F(\lambda_1, \lambda_2) \ge 0 > F(\lambda_1 + \mu_1, \lambda_2)$, with $F(\lambda_1 + \mu_1, \lambda_2) < 0$ if and only if

$$\lambda_2 - \mu_2 < \lambda_1 + \mu_1 \tag{14}$$

[14] To this purpose we use $G_1(\bar{b}_{1M}) = \lambda_1 + \mu_1$ and, from Lemma 1, $G_1(\bar{b}_{2M}) = \lambda_2 + \mu_2$.

[15] Notice that $F(\rho_1, \rho_2) = 0$ implies that $\bar{b}_H$ in (8) can be written as $v_H - (\rho_1 + \lambda_2 + \mu_2)\Delta$.

The unique equilibrium is P_{1MH} if $F(\lambda_1+\mu_1,\lambda_2)\geq 0$.

By Proposition 1, there exist three different equilibrium regimes, (10)-(12), and the regime which applies is determined by the parameter values through the sign of $F(\lambda_1,\lambda_2)$ and of $F(\lambda_1+\mu_1,\lambda_2)$.[16] For instance, when $F(\lambda_1,\lambda_2)>0$ the utility of type 2_H from bidding $\bar{b}_{2M}$ is lower than the utility from $\bar{b}_H$, which is inconsistent with equilibrium. Equality is achieved by increasing ρ_1 above λ_1, as this increases the probability that bidder 1 bids less than $\bar{b}_{2M}$ (and in particular the probability that 1 bids v_L), hence increases 2_H's utility from $\bar{b}_{2M}$, but also increases $\bar{b}_H$, hence decreasing the utility from $\bar{b}_H$.[17] In particular, ρ_1 is determined by solving $F(\rho_1,\lambda_2)=0$ if a solution to this equation exists in the interval $(\lambda_1,\lambda_1+\mu_1)$, and then P_{1M} in (11) is identified.[18] But in some cases $F(\lambda_1+\mu_1,\lambda_2)\geq 0$, that is $F(\rho_1,\lambda_2)=0$ has no solution smaller than $\lambda_1+\mu_1$. Then $\rho_1\geq\lambda_1+\mu_1$, that is type 1_M bids v_L with probability 1 and $\bar{b}_{1M}=v_L$; this implies that (7) is violated and the derivation of (9) based on (8) does not apply when $\rho_1\geq\lambda_1+\mu_1$. However, the value of ρ_1 is still determined by the condition that 2_H is indifferent between bidding $\bar{b}_{2M}$ or $\bar{b}_H$, and then P_{1MH} in (12) is identified (see the proof of Proposition 1). An important feature of P_{1MH} is that bidding is not affected by λ_1,μ_1.[19]

Figures 1a,1b below provide a graphical illustration of Proposition 1 by fixing λ_2,μ_2, and representing the space of (λ_1,μ_1) which satisfy (1), that is the triangle with bold edges and vertices $(0,0)$, $(\lambda_2+\mu_2,0)$, $(0,\lambda_2+\mu_2)$; the point $(\lambda_1,\mu_1)=(\lambda_2,\mu_2)$ is on the hypothenuse of this triangle. Figure 1a refers to the case with $\lambda_2\leq\mu_2$, which makes (14) satisfied for each (λ_1,μ_1), whereas Figure 1b refers to the case of $\lambda_2>\mu_2$, which allows for the existence of (λ_1,μ_1) close to $(0,0)$ which violate (14). The equality $F(\lambda_1,\lambda_2)=0$ is equivalent to (13) written with equality and it holds when, in Figure 1a, (λ_1,μ_1) is on the curve C connecting point $(0,0)$ to point (λ_2,μ_2); in Figure 1b, $F(\lambda_1,\lambda_2)=0$ holds when (λ_1,μ_1) is on the curve C connecting

[16] Proposition 1 assumes that (1) is satisfied, but it is simple to adapt the analysis to the case in which (1) does not hold.

[17] This occurs because increasing ρ_1 lowers $G_2(\bar{b}_{1M})=\frac{\rho_2}{\rho_1}(\lambda_1+\mu_1)$, which lowers type 1_H's utility from bidding $\bar{b}_{1M}$. Since type 1_H must be indifferent between bidding $\bar{b}_{1M}$ and bidding $\bar{b}_H$, it follows that $\bar{b}_H$ increases.

[18] When $F(\lambda_1,\lambda_2)<0$, a similar argument applies to explain why $\rho_2>\lambda_2$, based on comparing the utility of type 1_H from bidding $\bar{b}_{1M}$ and the utility from bidding $\bar{b}_H$.

[19] We remark that $\rho_1>\lambda_1+\mu_1$ means that type 1_H bids v_L with positive probability. This may occur only when $\lambda_2>\mu_2$ because 1_H's utility from bidding v_L is $2\lambda_2\Delta$, and by bidding v_M, 1_H wins with probability greater than $\lambda_2+\mu_2$, earning utility greater than $(\lambda_2+\mu_2)\Delta$. Thus $\lambda_2\leq\mu_2$ makes the bid v_L less profitable than the bid v_M and rules out that 1_H bids v_L.

point $(\lambda_2 - \mu_2, 0)$ to point (λ_2, μ_2):

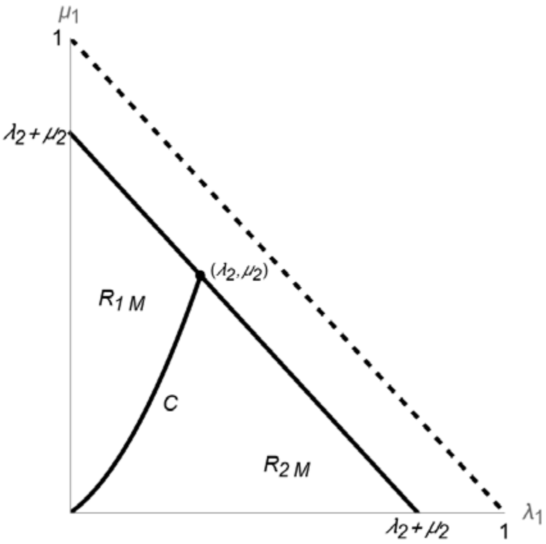

Figure 1a: The regions R_{1M}, R_{2M} when $\lambda_2 \leq \mu_2$

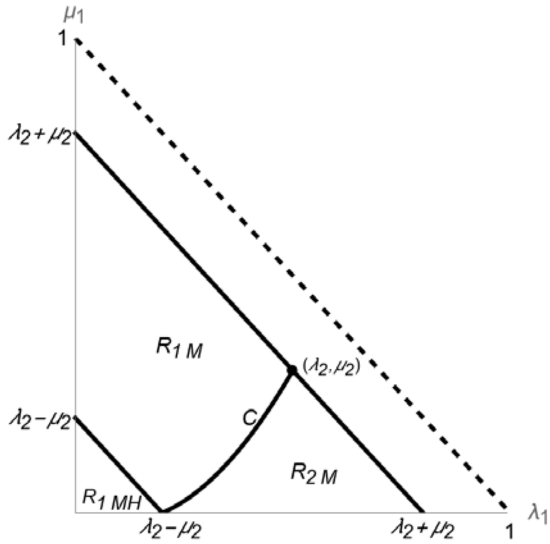

Figure 1b: The regions R_{1M}, R_{2M}, R_{1MH} when $\lambda_2 > \mu_2$

Region R_{2M} in either figure is the set of (λ_1, μ_1) for which (13) is satisfied, hence P_{2M} is the equilibrium when $(\lambda_1, \mu_1) \in R_{2M}$. Region R_{1M} is the region in which the equilibrium is P_{1M}, and P_{1MH} is the equilibrium when $(\lambda_1, \mu_1) \in R_{1MH}$ (R_{1MH} is empty in Figure 1a). If (λ_1, μ_1) lies on C, then P_{1M} (or equivalently, P_{2M}) is the equilibrium, with $(\rho_1, \rho_2) = (\lambda_1, \lambda_2)$, that is only types 1_L and 2_L bid v_L.

After $\rho_1, \rho_2, \bar{b}_H$ are determined, the rent of each bidder type follows from (3)-(6) and depends on whether (λ_1, μ_1) belongs to region R_{2M}, or to R_{1M}, or to R_{1MH}: see the proof of Proposition 1.

The expected revenue R^F is the expectation of the highest bid, which is equal to v_L with probability $\rho_1\rho_2$ and has c.d.f. $G_1(b)G_2(b)$ for $b \in (v_L, \bar{b}_H]$, with $G_1(b)$ and $G_2(b)$ determined by (3)-(6). Hence

$$R^F = \rho_1\rho_2 v_L + \int_{v_L}^{\bar{b}_H} bd\left(G_1(b)G_2(b)\right)$$

3.1.3 The effects of asymmetry on bidding in the FPA

In this subsection we describe the effects of the asymmetry in the distributions of values on the equilibrium bidding in the FPA. In particular, we compare bidding under asymmetry with bidding under symmetry, and to this purpose we indicate with G_1^{sym} the c.d.f. of bids of either bidder in the symmetric setting in which v_1, v_2 are i.i.d. and $\Pr\{v_1 = v_L\} = \Pr\{v_2 = v_L\} = \lambda_1$, $\Pr\{v_1 = v_M\} = \Pr\{v_2 = v_M\} = \mu_1$. Likewise, with G_2^{sym} we denote the c.d.f. of bids of either bidder when v_1, v_2 are i.i.d. and $\Pr\{v_1 = v_L\} = \Pr\{v_2 = v_L\} = \lambda_2$, $\Pr\{v_1 = v_M\} = \Pr\{v_2 = v_M\} = \mu_2$. Proposition 1 reveals that, for $i = 1, 2$,

$$G_i^{\text{sym}}(b) = \begin{cases} \frac{\lambda_i \Delta}{v_M - b} & \text{for each } b \in [v_L, \bar{\beta}_{iM}] \\ \frac{v_H - \bar{\beta}_{iH}}{v_H - b} & \text{for each } b \in (\bar{\beta}_{iM}, \bar{\beta}_{iH}] \end{cases} \quad \text{with} \quad \bar{\beta}_{iM} = v_M - \frac{\lambda_i}{\lambda_i + \mu_i}\Delta, \quad \bar{\beta}_{iH} = v_H - (2\lambda_i + \mu_i)\Delta$$

The following proposition describes how the asymmetry affects bidding in the FPA.

Proposition 2 Suppose that (1) holds strictly. Then
(i) $G_{2M}(b) < G_{1M}(b)$ for each $b \in [v_L, \bar{b}_{1M}]$ if and only if $(\lambda_1, \mu_1) \in R_{1M} \cup R_{1MH}$;
(ii) $G_{2H}(b) < G_{1H}(b)$ for each $b \in [\bar{b}_{2M}, \bar{b}_H)$;

(iii) $G_1(b) \leq G_2(b)$ for each $b \in [v_L, \bar{b}_H]$;
(iv) $G_1(b) \leq G_2^{\text{sym}}(b)$ for each $b \in [v_L, \bar{\beta}_{2H}]$ if and only if $\lambda_1 \leq \lambda_2$; $G_2^{\text{sym}}(b) < G_1(b)$ for each $b \in [v_L, \bar{b}_H]$ if and only if $\lambda_2 < \lambda_1$;
(v) $G_1^{\text{sym}}(b) \leq G_2(b)$ for each $b \in [v_L, \bar{b}_H]$;
(vi) $G_1^{\text{sym}}(b) \leq G_1(b)$ for each $b \in [v_L, \bar{b}_H]$;
(vii) $G_2(b) \leq G_2^{\text{sym}}(b)$ for each $b \in [v_L, \bar{\beta}_{2H}]$ if and only if $(\lambda_1, \mu_1) \in R_{1M} \cup R_{1MH}$; $G_2^{\text{sym}}(b) \leq G_2(b)$ for each $b \in [v_L, \bar{b}_H]$ if and only if $\lambda_1 \geq \lambda_2$.

A well known result in the literature refers to a setting with two bidders, one of whom is ex ante stronger and the other is ex ante weaker, in the sense that the c.d.f. for the strong bidder's value dominates the c.d.f. for the weak bidder's value in terms of conditional stochastic dominance.[20] That is, the former c.d.f. first order stochastically dominates the latter c.d.f., conditional on considering values not greater than a given v, for an arbitrary v. This condition is proved to imply that the weak bidder bids more aggressively than the strong bidder, that is for a same value for the object, the weak bidder's bid is higher than the strong bidder's bid.[21]

Proposition 2(i-ii) proves a similar result after observing that in our setting, given (1), if one bidder is stronger than the other then it is bidder 1. In particular, Proposition 2(i) shows that given (1) strictly satisfied, the condition $(\lambda_1, \mu_1) \in R_{1M} \cup R_{1MH}$ is necessary and sufficient for type 2_M to bid more aggressively than type 1_M. When instead $(\lambda_1, \mu_1) \in R_{2M}$, we find that $G_{1M}(b) < G_{2M}(b)$ for b close to v_L because $\rho_2 > \lambda_2$, hence type 2_M bids v_L with positive probability, unlike type 1_M.[22] Proposition 2(ii) shows that (1) suffices to prove that type 2_H bids more aggressively than type 1_H – a result analogous to Proposition 1 of Fibich, Gavious, Sela (2002). In order to compare the assumption of Proposition 2(i-ii) with conditional stochastic dominance, we notice that in our setting the latter dominance holds if and only if the distribution of v_1 first order stochastically dominates the distribution of v_2 (that is, if $\lambda_1 \leq \lambda_2$), and moreover

$$\Pr\{v_1 = v_L | v_1 \leq v_M\} < \Pr\{v_2 = v_L | v_2 \leq v_M\}$$

which reduces to

$$\frac{\lambda_1}{\lambda_1 + \mu_1} < \frac{\lambda_2}{\lambda_2 + \mu_2} \tag{15}$$

Given (λ_2, μ_2), a graphical interpretation of (15) is immediate: (15) holds if and only if (λ_1, μ_1) in Figure 1 lies above the segment connecting $(0, 0)$ to (λ_2, μ_2). This implies that $(\lambda_1, \mu_1) \in R_{1M} \cup R_{1MH}$,[23] thus our Proposition 2(i-ii) relies on a condition weaker than conditional stochastic dominance.

Proposition 2(iii) proves that even though bidder 2 may bid more aggressively than bidder 1, the overall bid distribution of bidder 1 is stronger than the overall bid distribution of bidder 2. Li and Riley (2007) prove this result assuming that the c.d.f. for the value of the strong bidder first order stochastically dominates the c.d.f. for the value of the weak bidder, but we notice that Proposition 2(iii) relies on a condition weaker than first order stochastic dominance, as $\lambda_1 \leq \lambda_2$ is not needed.

Proposition 2(iv-vii) provides results about how asymmetry affects bidding in the FPA with respect to a symmetric setting. In particular, the first inequality in Proposition 2(iv) considers a bidder facing a weak opponent and establishes that a strong bidder bids more aggressively than a weak bidder against a weak

[20] See Lebrun (1998), Maskin and Riley (2000a), Li and Riley (2007), Kirkegaard (2009). Lebrun (1998) allows for an arbitrary number of ex ante strong bidders and an arbitrary number of ex ante weak bidders.

[21] Precisely, the papers mentioned in the previous footnote consider settings with continuously distributed values in which each bidder type plays a pure strategy.

[22] But $G_{2M}(b) < G_{1M}(b)$ for b close to $\bar{b}_{1M}$ as $G_{1M}(\bar{b}_{1M}) = 1$ and $\bar{b}_{1M} < \bar{b}_{2M}$.

[23] Hence the equilibrium in the FPA is P_{1M} or P_{1MH}.

opponent; equivalently, a weak bidder faces a more aggressive bid distribution from his opponent if the latter is strong rather than weak. Proposition 2(v) establishes an analogous property for a bidder facing a strong opponent. Proposition 2(vi) considers a strong bidder and claims that such a bidder bids more aggressively when facing a strong rather than a weak opponent. Likewise, the first inequality in Proposition 2(vii) establishes that a weak bidder bids more aggressively when facing a strong opponent rather than a weak opponent when $(\lambda_1, \mu_1) \in R_{1M} \cup R_{1MH}$, but not when $(\lambda_1, \mu_1) \in R_{2M}$. In fact, the opposite property holds if $\lambda_1 \geq \lambda_2$, that is a weak bidder is less aggressive against a strong opponent than against a weak opponent if $\lambda_1 \geq \lambda_2$.

Maskin and Riley (2000a) assume first order or conditional stochastic dominance to establish the inequalities in Proposition 2(iv-vii) [for Proposition 2(iv), 2(vii) Maskin and Riley (2000a) prove the first inequality in each statement]. Conversely, Proposition 2(iv-vii) determine a less restrictive necessary and sufficient condition for either considered property to hold, and in some cases for the reverse property to hold.

3.2 Equilibrium bidding in the second price auction

In the second price auction, SPA henceforth, for each bidder it is weakly dominant to bid the own valuation. We use u_{ij}^S to denote the expected utility of type i_j, for $j = L, M, H$, $i = 1, 2$. Hence $u_{1L}^S = u_{2L}^S = 0$, that is, type 1_L and type 2_L both have utility zero, and

$$u_{1M}^S = \lambda_2 \Delta, \qquad u_{1H}^S = (2\lambda_2 + \mu_2)\Delta, \qquad u_{2M}^S = \lambda_1 \Delta, \qquad u_{2H}^S = (2\lambda_1 + \mu_1)\Delta \tag{16}$$

The seller's expected revenue R^S is the expectation of the second highest valuation, that is $R^S = v_L + (\mu_1\mu_2 + \mu_1\eta_2 + \eta_1\mu_2)\Delta + 2\eta_1\eta_2\Delta$, and after simple manipulation it can be written as

$$R^S = v_L + ((2 - 2\lambda_2 - \mu_2)(1 - \lambda_1) - (1 - \lambda_2 - \mu_2)\,\mu_1)\,\Delta \tag{17}$$

4 Comparison between the FPA and the SPA

In this section we compare the expected revenue in the FPA with the expected revenue in the SPA. To this purpose, it is useful to define

$$U^F = \mu_1 u_{1M}^F + \eta_1 u_{1H}^F + \mu_2 u_{2M}^F + \eta_2 u_{2H}^F \qquad \text{and} \qquad U^S = \mu_1 u_{1M}^S + \eta_1 u_{1H}^S + \mu_2 u_{2M}^S + \eta_2 u_{2H}^S$$

as the total bidders' expected utility under the FPA and under the SPA, respectively.

The SPA always allocates the object to a bidder with the highest value, whereas the FPA implements an inefficient allocation with positive probability when (1) holds strictly because then $\bar{b}_{1M} < \bar{b}_{2M}$ and type 2_M wins with positive probability when facing type 1_H.[24] Therefore social welfare is greater in the SPA than in the FPA, and if we establish that $U^F \geq U^S$ holds then it follows that $R^S > R^F$.

Example: Revenue ranking for the case of binary support The comparison between U^F and U^S yields an immediate conclusion in the setting with binary support with $\mu_1 = 0$, $\mu_2 = 0$ and $\lambda_1 < \lambda_2$. The equilibrium in the FPA described at the end of Subsection 3.1.1 coincides with P_{1MH} in (12) with $\rho_1 = \lambda_2$, $\rho_2 = \lambda_2$. As a result, types $1_L, 2_L, 1_H$ earn the same utility in the FPA as in the SPA, but type 2_H's utility is higher in the FPA than in the SPA, $2\lambda_2\Delta$ rather than $2\lambda_1\Delta$. Therefore $U^F > U^S$ and $R^S > R^F$.

In the following we show that the setting with three types leads to significantly different results and we illustrate the source of the difference.

[24] Conversely, $\bar{b}_{1M} = \bar{b}_{2M}$ holds when (1) is satisfied with equality, and then the FPA allocates the object efficiently.

4.1 Comparison of rents

It is immediate from (2) and (16) that both type 1_M and 2_M weakly prefer the FPA, that is $u_{1M}^F \geq u_{1M}^S$ and $u_{2M}^F \geq u_{2M}^S$, since $\rho_2 \geq \lambda_2$ and $\rho_1 \geq \lambda_1$. This occurs because in the FPA type 1_M or type 2_M (unless (13) is an equality) bids v_L with positive probability, which makes type 2_M or type 1_M better off than in the SPA.

The same preference holds for type 2_H, that is $u_{2H}^F \geq u_{2H}^S$, because u_{2H}^F is given by type 2_H's expected utility from bidding $\bar{b}_{2M}$, that is $(v_H - \bar{b}_{2M})G_1(\bar{b}_{2M})$, which reduces to $(v_M - \bar{b}_{2M} + \Delta)G_1(\bar{b}_{2M})$, and finally to $u_{2M}^F + (\lambda_2 + \mu_2)\Delta$ since $G_1(\bar{b}_{2M}) = \lambda_2 + \mu_2$ by Lemma 1. Using (2) we see that $u_{2M}^F + (\lambda_2 + \mu_2)\Delta$ is not less than $(\lambda_1 + \lambda_2 + \mu_2)\Delta$, which is at least as large as u_{2H}^S in (16) because of (1). Basically, in the FPA type 2_H benefits from the fact that $\bar{b}_{1M} < \bar{b}_{2M}$, that is type 1_H submits a bid below $\bar{b}_{2M}$ with positive probability. Hence type 1_H loses with positive probability against $\bar{b}_{2M}$, the highest bid submitted by type 2_M, that is when bidding $\bar{b}_{2M}$, type 2_H beats types $1_L, 1_M$ for sure and type 1_H with positive probability. Thus 2_H wins with probability greater than $\lambda_1 + \mu_1$. Conversely, under the SPA type 2_H wins and earns a positive utility only when facing type 1_L or 1_M, that is with probability $\lambda_1 + \mu_1$.

Matters are different for type 1_H, because $u_{1H}^F = u_{2H}^F$ but $u_{1H}^S = (2\lambda_2 + \mu_2)\Delta$ may be higher than $u_{2H}^S = (2\lambda_1 + \mu_1)\Delta$, and indeed $u_{1H}^S > u_{1H}^F$ in some cases. For instance, this occurs when $(\lambda_1, \mu_1) \in R_{2M}$ and $\lambda_1 < \lambda_2$ because we know from above that $u_{1H}^F = u_{2H}^F = u_{2M}^F + (\lambda_2 + \mu_2)\Delta = (\lambda_1 + \lambda_2 + \mu_2)\Delta$ (as $\rho_1 = \lambda_1$), hence $u_{1H}^S > u_{1H}^F$. Next lemma shows that $u_{1H}^F < u_{1H}^S$ when $\lambda_1 < \lambda_2$ and (1) is strictly satisfied.

Lemma 2 Types $1_M, 2_M, 2_H$ all weakly prefer the FPA to the SPA. Type 1_H prefers the FPA if $\lambda_1 > \lambda_2$; type 1_H is indifferent between the two auctions if $\lambda_1 = \lambda_2$, or if $\lambda_1 < \lambda_2$ and (1) holds with equality; type 1_H prefers the SPA if $\lambda_1 < \lambda_2$ and (1) holds strictly.

By Lemma 2, only type 1_H may prefer the SPA to the FPA, hence it is intuitive that often $U^F \geq U^S$. In particular, when $\lambda_1 \geq \lambda_2$ each bidder type weakly prefers the FPA to the SPA, hence $U^F > U^S$ if $\lambda_1 \geq \lambda_2$.[25] Next proposition provides an alternative sufficient condition for $U^F > U^S$.

Proposition 3 Given (1) and $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$, either of the following two conditions implies $U^F > U^S$, hence $R^S > R^F$: (i) $\lambda_1 \geq \lambda_2$; (ii) $\lambda_1 < \lambda_2$ and μ_1 is large, that is (1) holds with equality or with approximate equality.

About condition (ii) in Proposition 3, notice that Lemma 2 establishes $u_{1H}^F = u_{1H}^S$ when $\lambda_1 < \lambda_2$ and (1) is satisfied with equality, that is when $\mu_1 = \lambda_2 + \mu_2 - \lambda_1$; hence each type weakly prefers the FPA and since $(\lambda_1, \mu_1) \in R_{1M}$, we have that $\rho_1 > \lambda_1$, hence type 2_M strictly prefers the FPA and $U^F > U^S$. In the proof to Proposition 3 we show that $U^F > U^S$ still holds if μ_1 is slightly reduced below $\lambda_2 + \mu_2 - \lambda_1$. The basic insight is that a reduction in μ_1 lowers u_{2H}^S in (16), and even though it may reduce also u_{1H}^F and u_{2H}^F, and increase the weight η_1 of $u_{1H}^F - u_{1H}^S \leq 0$ in $U^F - U^S$, the latter effects are relatively less important than the reduction in u_{2H}^S when starting from $\mu_1 = \lambda_2 + \mu_2 - \lambda_1$. In particular, Proposition 3(ii) implies $U^F > U^S$ if (λ_1, μ_1) is close to (λ_2, μ_2). Therefore $R^S > R^F$ when the asymmetry is small.[26]

4.2 Comparison of revenues

4.2.1 General comparison

In this subsection we compare R^F with R^S directly, without resorting to the comparison of rents. But since Proposition 3 reveals that $R^F < R^S$ if $\lambda_1 \geq \lambda_2$ and/or if (1) holds with equality, we consider the case in

[25] Since we are considering $(\lambda_1, \mu_1) \in R_{2M}$, we have that $\rho_2 > \lambda_2$. Hence type 1_M strictly prefers the FPA and $U^F > U^S$.

[26] Gavious and Minchuk (2013) prove that no general ranking result holds for small asymmetries around the uniform distribution. Conversely, in our setting small asymmetries always favor the SPA.

which $\lambda_1 < \lambda_2$ and (1) holds with strict inequality. This means that the c.d.f. of v_1 first order stochastically dominates the c.d.f. of v_2.

We begin by observing that (17) reveals that R^S is strictly decreasing in λ_1 and in μ_1, which is intuitive as an increase in λ_1 (in μ_1) increases the probability that bidder 1 has value v_L (has value v_M) and decreases the probability $\eta_1 = 1 - \lambda_1 - \mu_1$ that bidder 1 has value v_H. Hence it worsens the bid distribution of bidder 1. Moreover,

$$\frac{\partial R^S}{\partial \lambda_1} < \frac{\partial R^S}{\partial \mu_1} < 0 \tag{18}$$

holds since $v_L < v_M$. Hence an increase in λ_1 decreases R^S more than an equal increase in μ_1.

The expression for R^F, given in the proof of Proposition 1, depends on whether (λ_1, μ_1) belongs to R_{1M}, or to R_{2M}, or to R_{1MH}, and is not as simple as (17). Then, instead of trying to determine the precise set of solutions to the inequality $R^F > R^S$, for given (λ_2, μ_2) we inquire whether there exist (λ_1, μ_1) such that $R^F > R^S$, or if instead $R^F \leq R^S$ for each (λ_1, μ_1).

Next proposition provides an answer after distinguishing the case of $\lambda_2 \leq \mu_2$ from the case of $\lambda_2 > \mu_2$ because when $\lambda_2 \leq \mu_2$ there exist two equilibrium regimes, one applying when $(\lambda_1, \mu_1) \in R_{1M}$, the other when $(\lambda_1, \mu_1) \in R_{2M}$: see Figure 1a. Conversely, when $\lambda_2 > \mu_2$ there is a third equilibrium regime, which applies when λ_1, μ_1 are small, that is when $(\lambda_1, \mu_1) \in R_{1MH}$: see Figure 1b.

Proposition 4(i) Suppose that $\lambda_2 \leq \mu_2$. If $(\lambda_2 + \mu_2)^2 \leq \mu_2$, then $R^F \leq R^S$ for each (λ_1, μ_1) which satisfies (1). Conversely, if

$$(\lambda_2 + \mu_2)^2 - \mu_2 > 0 \tag{19}$$

then $R^F > R^S$ for (λ_1, μ_1) close to $(0, 0)$, and there exists $\lambda_1^* \in (0, \lambda_2)$ such that $(\lambda_1, \mu_1) \in$ C satisfies $R^F > R^S$ if and only if $\lambda_1 < \lambda_1^*$.
(ii) Suppose that $\lambda_2 > \mu_2$. If

$$3\lambda_2 + \mu_2 - 1 - 2\frac{\lambda_2}{\mu_2}(\lambda_2 - \mu_2) \ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right) > 0 \tag{20}$$

then $R^F > R^S$ for (λ_1, μ_1) close to $(\lambda_2 - \mu_2, 0)$, and there exists $\lambda_1^* \in (\lambda_2 - \mu_2, \lambda_2)$ such that $(\lambda_1, \mu_1) \in$ C satisfies $R^F > R^S$ if and only if $\lambda_1 < \lambda_1^*$. If (20) is violated, then $R^F \leq R^S$ for each (λ_1, μ_1) satisfying (1).
(iii) Regardless of whether $\lambda_2 \leq \mu_2$ or $\lambda_2 > \mu_2$, the inequality $R^F \leq R^S$ holds for each (λ_1, μ_1) if $\lambda_2 + \mu_2 \leq \frac{1}{2}$.

The case of $\lambda_2 \leq \mu_2$ In the following, for ease of language, instead of (λ_1, μ_1) close to $(0, 0)$ we write $(\lambda_1, \mu_1) = (0, 0)$. When $\lambda_2 \leq \mu_2$, Proposition 4(i) establishes that there exist (λ_1, μ_1) such that $R^F > R^S$ if and only if (19) is satisfied, and in such case $R^F > R^S$ holds for $(\lambda_1, \mu_1) = (0, 0)$. In other words, if $R^F \leq R^S$ for $(\lambda_1, \mu_1) = (0, 0)$ then $R^F \leq R^S$ for each (λ_1, μ_1). In the proof of Proposition 4 we show that $\eta_2 \geq \frac{1}{2}$ makes (19) violated, hence $\eta_2 \geq \frac{1}{2}$, or equivalently $\lambda_2 + \mu_2 \leq \frac{1}{2}$, implies $R^F \leq R^S$ for each (λ_1, μ_1), as Proposition 4(iii) states. Conversely, (19) may hold when $\lambda_2 + \mu_2$ is greater than $\frac{1}{2}$, and in particular it holds if, given any $\mu_2 > \frac{1}{4}$,[27] λ_2 is sufficiently large, given that $\lambda_2 \leq \mu_2$: see Figure 2 below.

Next property helps to make sense of Proposition 4(i).

C-property If $R^F - R^S > 0$ for some $(\lambda_1, \mu_1) \notin$ C, then there exists at least one $(\lambda_1, \mu_1) \in$ C such that $R^F - R^S > 0$.

[27] Since $\lambda_2 + \mu_2 > \frac{1}{2}$ and $\mu_2 \geq \lambda_2$, it follows that $\mu_2 > \frac{1}{4}$.

The proof of this property exploits the features of P_{1M}, P_{2M} in (11), (10) to show that starting from any $(\lambda_1', \mu_1') \notin$ C, it is possible to find $(\lambda_1'', \mu_1'') \in$ C such that $R^F - R^S$ is greater when $(\lambda_1, \mu_1) = (\lambda_1'', \mu_1'')$ than when $(\lambda_1, \mu_1) = (\lambda_1', \mu_1')$.

The C-property implies that in order to establish the existence of (λ_1, μ_1) such that $R^F - R^S > 0$, we can focus on $(\lambda_1, \mu_1) \in$ C, that is we search for $(\lambda_1, \mu_1) \in$ C such that $R^F > R^S$. Proposition 3 implies that $R^F < R^S$ if $(\lambda_1, \mu_1) \in$ C is close to (λ_2, μ_2), as then $U^F > U^S$. At the other extreme of C, that is at $(\lambda_1, \mu_1) = (0, 0)$, we find that $R^F - R^S$ is given by the left hand side of (19), which is positive if and only if λ_2 is sufficiently large, given $\lambda_2 \leq \mu_2$. We notice that $(\lambda_1, \mu_1) = (0, 0)$ induces the most aggressive bidding by both bidders in the FPA, given $\lambda_2 \leq \mu_2$, because from (3)-(6) and (8) it follows that G_1, G_2 are most aggressive the lower are ρ_1, ρ_2, and $\rho_1 \geq \lambda_1, \rho_2 \geq \lambda_2$ by (2). When $(\lambda_1, \mu_1) = (0, 0)$, we find $\rho_1 = 0$, $\rho_2 = \lambda_2$, the lowest possible values for ρ_1, ρ_2 given λ_2, μ_2. But we stress that $(\lambda_1, \mu_1) = (0, 0)$ alone is not sufficient for $R^F > R^S$ to hold: $(\lambda_1, \mu_1) = (0, 0)$ induces the most aggressive bidding by bidder 1 also in the SPA,[28] and the sign of $R^F - R^S$ when $(\lambda_1, \mu_1) = (0, 0)$ is determined by whether (19) is satisfied. We illustrate below why $R^F > R^S$ holds when (19) holds, based on the resulting bidding in the FPA.

First notice that when λ_1, μ_1 are about 0, bidder 1 almost certainly has value v_H and there are three relevant states of the world: (v_1, v_2) equal to (v_H, v_L), or equal to (v_H, v_M), or equal to (v_H, v_H). From (8) we see that $\bar{b}_{2M}$ is close to v_M, and (5) implies that $G_1(b)$ is close to 0 for each $b < v_M$, that is in the limit as $(\lambda_1, \mu_1) \to (0, 0)$, G_1 places all the probability on bids not smaller than v_M. In other words, bidder 1 (i.e., type 1_H) bids at least v_M;[29] hence the revenue under the FPA is greater than v_M. Conversely, under the SPA the revenue coincides with v_2 in each of the states mentioned above as $\min\{v_1, v_2\} = \min\{v_H, v_2\} = v_2$. Hence the revenue is greater in the FPA in the states $(v_1, v_2) = (v_H, v_L)$ and $(v_1, v_2) = (v_H, v_M)$, but it is greater in the SPA when $(v_1, v_2) = (v_H, v_H)$. This makes it intuitive that $R^F > R^S$ if λ_2 is large: the greater is λ_2, the greater is the probability of the state (v_H, v_L), in which the FPA has a higher revenue, and the lower is the probability of the state (v_H, v_H), in which the SPA has a greater revenue. For instance, $R^F > R^S$ when λ_2 is close to $1 - \mu_2$ because then (v_H, v_H), the only state in which the SPA is superior to the FPA, has probability about zero.[30] Conversely, λ_2 close to zero makes almost irrelevant the state (v_H, v_L) in which the FPA is superior to the SPA, and the expectation over the two remaining states, (v_H, v_M) and (v_H, v_H), yields $R^S > R^F$ because R^S is about $v_M + (1 - \mu_2)\Delta$, which coincides with the highest submitted bid in the FPA, $\bar{b}_H$ in (8).[31]

We have considered above only $(\lambda_1, \mu_1) \in$ C close to (λ_2, μ_2) or close to $(0, 0)$, neglecting intermediate $(\lambda_1, \mu_1) \in$ C, because we can prove that $R^F \leq R^S$ for $(\lambda_1, \mu_1) = (0, 0)$ implies $R^F \leq R^S$ for each other (λ_1, μ_1) in C. Precisely, $(\lambda_1, \mu_1) \in$ C implies $\mu_1 = \frac{1}{\lambda_2}\lambda_1^2 + \frac{\mu_2 - \lambda_2}{\lambda_2}\lambda_1$: see (13). Therefore, for $(\lambda_1, \mu_1) \in$ C, $R^F - R^S$ can be viewed as a function of λ_1 alone, which we write as $R^{F-S}(\lambda_1)$, for $\lambda_1 \in [0, \lambda_2]$. In the proof of Proposition 4 we show that R^{F-S} is a strictly convex function of λ_1, and we know that $R^{F-S}(\lambda_2) = 0$ because $\lambda_1 = \lambda_2$ makes v_1, v_2 identically distributed. Hence if $R^{F-S}(0) \leq 0$ – that is if (19) is violated – then $R^{F-S}(\lambda_1) < 0$ for each $\lambda_1 \in (0, \lambda_2)$, that is $R^F - R^S < 0$ for each $(\lambda_1, \mu_1) \in$ C. If instead $R^{F-S}(0) > 0$ – that is if (19) holds – then $(\lambda_1, \mu_1) = (0, 0)$ maximizes $R^F - R^S$ and there exists $\lambda_1^* \in (0, \lambda_2)$ such that $R^{F-S}(\lambda_1) > 0$ for $\lambda_1 \in (0, \lambda_1^*)$ but $R^{F-S}(\lambda_1) < 0$ for $\lambda_1 \in (\lambda_1^*, \lambda_2)$.

[28] The bidding of bidder 2 in the SPA is not affected by (λ_1, μ_1).

[29] This is quite intuitive as $\rho_1 = 0$ implies $u_{2M}^F = 0$, which requires that type 2_M has no possibility to win the auction with a bid lower than v_M. This occurs only if bidder 1 bids at least v_M with probability 1.

[30] Notice that λ_2 close to $1 - \mu_2$ requires $\mu_2 \geq \frac{1}{2}$, because if $\mu_2 < \frac{1}{2}$ then $\lambda_2 \leq \mu_2$ implies $\lambda_2 < \frac{1}{2}$ and $1 - \mu_2 > \frac{1}{2}$. If instead $\mu_2 \in (\frac{1}{4}, \frac{1}{2}]$, then λ_2 is at most equal to μ_2, and in such case still $R^F > R^S$.

[31] In order to justify this claim we rely on footnote 15 to conclude that $\bar{b}_H$ in (8) can be written as $v_H - (\rho_1 + \lambda_2 + \mu_2)\Delta$, which is equal to $v_M + (1 - \mu_2)\Delta$ since $\rho_1 = \lambda_1 = 0$, $\lambda_2 = 0$.

The case of $\lambda_2 > \mu_2$ When $\lambda_2 > \mu_2$, Proposition 4(ii) establishes a result analogous to Proposition 4(i), that is if $R^F > R^S$ holds for some (λ_1, μ_1), then it holds for $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$, or equivalently if $R^F \leq R^S$ at $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$, then $R^F \leq R^S$ for each (λ_1, μ_1). In a sense, now $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$ plays the role $(\lambda_1, \mu_1) = (0, 0)$ plays when $\lambda_2 \leq \mu_2$. In order to see why, recall from Subsection 3.1.2 that given $\lambda_2 > \mu_2$, if (λ_1, μ_1) is close to $(0, 0)$ then (14) is violated and R^F is constant with respect to λ_1, μ_1, whereas R^S in (17) is decreasing in λ_1, μ_1. Hence $(\lambda_1, \mu_1) = (0, 0)$ is the maximum point for R^S and the minimum point for $R^F - R^S$ in R_{1MH}, whereas $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$ is the maximum point for $R^F - R^S$ in R_{1MH} by (18). Moreover, $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$ is a point of C and the C-property holds for $\lambda_2 > \mu_2$, that is if $R^F > R^S$ for some $(\lambda_1, \mu_1) \notin$ C then there exist $(\lambda_1, \mu_1) \in$ C such that $R^F > R^S$. Finally, $R^F - R^S$ is strictly convex along C, that is the function $R^{F-S}(\lambda_1)$, defined for $\lambda_1 \in [\lambda_2 - \mu_2, \lambda_2]$, is strictly convex with respect to λ_1. Thus if $R^{F-S}(\lambda_1) > 0$ for some $\lambda_1 \in (\lambda_2 - \mu_2, \lambda_2)$ then $R^{F-S}(\lambda_2 - \mu_2) > 0$; indeed, (20) coincides with the condition $R^{F-S}(\lambda_2 - \mu_2) > 0$.

We remark that $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$ is the distribution of v_1 which induces both bidders' most aggressive bidding in the FPA, given $\lambda_2 > \mu_2$, because when $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$ we have $\rho_2 = \lambda_2$ (this is the minimum value for ρ_2) and $\rho_1 = \lambda_2 - \mu_2$, and for each other (λ_1, μ_1), ρ_1 is greater than $\lambda_2 - \mu_2$. However, $R^F > R^S$ holds at $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$ if and only if (20) is satisfied, which is equivalent to $\lambda_2 + \mu_2 > \frac{1}{2}$ and μ_2 sufficiently large, given $\lambda_2 > \mu_2$. Figure 2 represents in grey the set of (λ_2, μ_2) which satisfy (19) when $\lambda_2 \leq \mu_2$, or (20) when $\lambda_2 > \mu_2$.

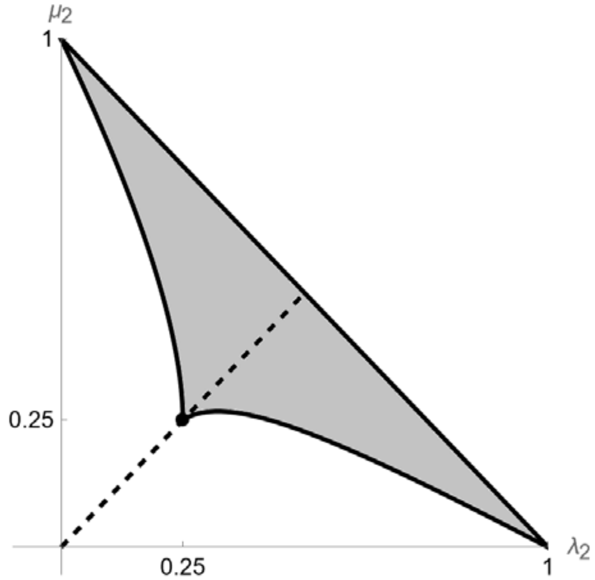

Figure 2: The set of (λ_2, μ_2) which satisfy (19) when $\lambda_2 \leq \mu_2$, or (20) when $\lambda_2 > \mu_2$

Inequality (20) is violated if $\lambda_2 + \mu_2 \leq \frac{1}{2}$, but if $\lambda_2 + \mu_2 > \frac{1}{2}$ then it is satisfied if μ_2 is sufficiently large, given $\lambda_2 > \mu_2$. For instance, if $\mu_2 = 1 - \lambda_2$ and $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$,[32] then there are four relevant states of the world: (v_1, v_2) equal to (v_L, v_L), or equal to (v_L, v_M), or equal to (v_H, v_L), or equal to (v_H, v_M). In all these states the revenue with the SPA is v_L, except in state (v_H, v_M) when it is v_M; hence

[32] Notice that $\mu_2 = 1 - \lambda_2$ requires $\lambda_2 > \frac{1}{2}$, since $\frac{1}{2} \geq \lambda_2 > \mu_2$ is inconsistent with $\mu_2 = 1 - \lambda_2$. In case that $\lambda_2 \in (\frac{1}{4}, \frac{1}{2})$, then $\lambda_2 > \mu_2$ imposes an upper bound on μ_2, and when μ_2 is close to the upper bound $R^F > R^S$ holds and we know from the case of $\lambda_2 \leq \mu_2$ that $R^F > R^S$ when $\lambda_2 = \mu_2$ and (λ_1, μ_1) is close to $(0, 0)$ (i.e., close to $(\lambda_2 - \mu_2, 0)$); hence by continuity $R^F > R^S$ holds also if μ_2 is just a bit smaller than λ_2.

$R^S = v_L + (1-\lambda_1)\mu_2\Delta$. For the FPA, the revenue is v_L in state (v_L, v_L), but in the other states mentioned above type 2_M and/or type 1_H submits a random bid with support $[v_L, v_L + 2\mu_2\Delta]$ because (8) reveals that $\bar{b}_{2M} = \bar{b}_H = v_L + 2\mu_2\Delta$ when $\lambda_2 + \mu_2 = 1$.[33] Moreover, the c.d.f.s for these bids, G_{2M} and G_{1H}, are convex, that is their densities are increasing. Hence the average bid of type 2_M and of type 1_H is no less than the midpoint of the support, $v_L + \mu_2\Delta$. Therefore R^F is greater than $v_L + (\lambda_1\mu_2 + 1 - \lambda_1)\,\mu_2\Delta$, which is greater than R^S identified above. Conversely, if μ_2 is about zero then $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$ is close to (λ_2, μ_2) and then a result similar to Proposition 3(ii) establishes that $U^F > U^S$, thus $R^S > R^F$.[34]

On the set of (λ_1, μ_1) such that $R^F > R^S$ When (λ_2, μ_2) is in the grey region in Figure 2, the set of (λ_1, μ_1) such that $R^F > R^S$ is non-empty, but as we explained above, a simple analytical characterization of this set is unavailable because the expression for R^F in the proof of Proposition 1 is complicated. Figures 3a, 3b below rely on numeric techniques to identify this set when $\lambda_2 = 0.4, \mu_2 = 0.5$ (Figure 3a), and when $\lambda_2 = 0.6, \mu_2 = 0.3$ (Figure 3b):

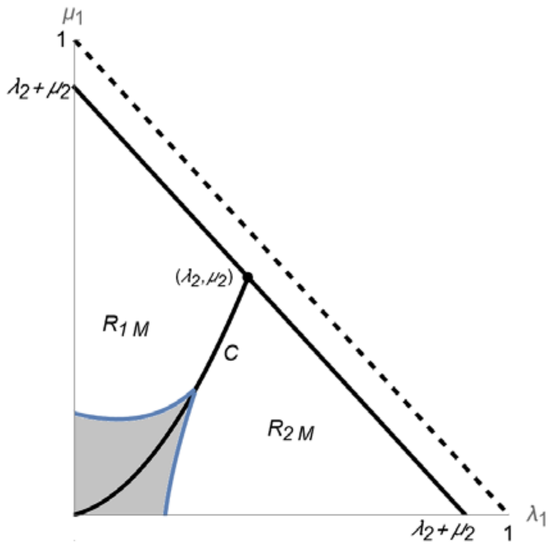

Figure 3a: The set of (λ_1, μ_1) such that $R^F > R^S$ when $\lambda_2 = 0.4, \mu_2 = 0.5$

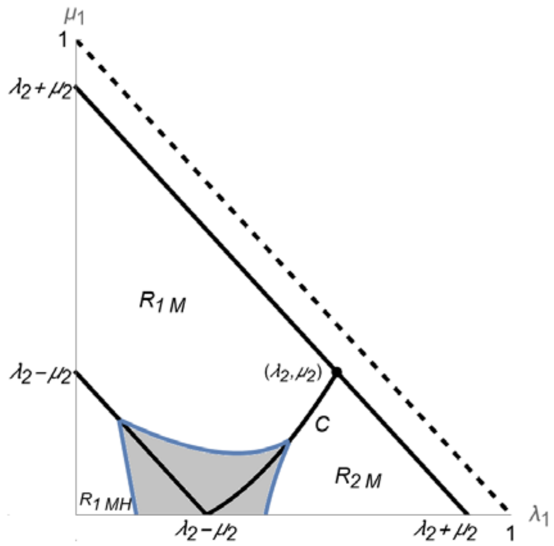

Figure 3b: The set of (λ_1, μ_1) such that $R^F > R^S$ when $\lambda_2 = 0.6, \mu_2 = 0.3$

Proposition 5 determines analytically a simple set, a trapezoid, which is included in the set of (λ_1, μ_1) which satisfy $R^F > R^S$.

Proposition 5(i) Suppose that $\lambda_2 \le \mu_2$ and (19) is satisfied. Let $\alpha = (\lambda_2+\mu_2)^2 - \mu_2 > 0$, and $\ell_1 = \frac{1}{2}\frac{\alpha^2}{(\lambda_2+\mu_2)^2}$, $m_1 = \frac{1}{\lambda_2}\ell_1^2 + \frac{\mu_2-\lambda_2}{\lambda_2}\ell_1$, hence $(\ell_1, m_1) \in$ C. The inequality $R^F > R^S$ holds for each (λ_1, μ_1) in the trapezoid with vertices $(0,0), (\ell_1, 0), (\ell_1, m_1), (0, \ell_1 + m_1)$.
(ii) Suppose that $\lambda_2 > \mu_2$ and (20) is satisfied. Let $\beta = \mu_2 + 3\lambda_2 - 1 - 2(\lambda_2 - \mu_2)\frac{\lambda_2}{\mu_2}\ln\frac{\lambda_2}{\lambda_2-\mu_2} > 0$, and $\ell_1 = \lambda_2 - \mu_2 + \frac{\mu_2\beta}{\lambda_2(1+2\ln\frac{\lambda_2}{\lambda_2-\mu_2})}$, $m_1 = \frac{1}{\lambda_2}\ell_1^2 + \frac{\mu_2-\lambda_2}{\lambda_2}\ell_1$, hence $(\ell_1, m_1) \in$ C. The inequality $R^F > R^S$ holds for each (λ_1, μ_1) in the trapezoid with vertices $(\gamma, 0), (\ell_1, 0), (\ell_1, m_1), (\gamma, \ell_1 + m_1 - \gamma)$, in which $\gamma = \max\{\lambda_2 - \mu_2 - m_1, 0\}$.

[33] In order to justify this claim we rely on footnote 15 to conclude that $\bar{b}_H$ in (8) can be written as $v_H - (\rho_1 + \lambda_2 + \mu_2)\Delta$, which is equal to $v_L + 2\mu_2\Delta$ since $\rho_1 = \lambda_1 = \lambda_2 - \mu_2$ and $\lambda_2 = 1 - \mu_2$.

[34] The result in Proposition 3(ii) does not apply because it assumes that (λ_2, μ_2) is fixed, whereas in the case we are considering μ_2 is not fixed.

The difference with the setting with binary support At the beginning of Section 4 we have remarked that $R^S > R^F$ when the support for each bidder's value is $\{v_L, v_H\}$, that is when $\mu_1 = \mu_2 = 0$, for each $\lambda_1 < \lambda_2$. Conversely, Proposition 4 shows that $R^F > R^S$ in some cases when the support is $\{v_L, v_M, v_H\}$.

In order to explain this difference, start from a symmetric setting with support $\{v_L, v_M, v_H\}$ and $0 < \mu_1 = \mu_2 < \lambda_1 = \lambda_2$.[35] Then consider a reduction in (λ_1, μ_1) along curve C, from (λ_2, μ_2) to $(\lambda_2 - \mu_2, 0)$: see Figure 1b. This reduces $\rho_1 = \lambda_1$ and leaves ρ_2 unchanged, hence improves bidding in the FPA and increases R^F. However, the reduction in (λ_1, μ_1) improves bidding also in the SPA and increases R^S, hence it is uncertain whether $R^F > R^S$ holds at $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$ – this is determined by whether (20) is satisfied. The main point is that the considered improvement in the distribution of v_1 increases both R^F and R^S.

Matters are different when the support is binary because $\mu_2 = 0$ implies that the set of (λ_1, μ_1) which satisfy (1) consists entirely of region R_{1MH} (regions R_{1M}, R_{2M} are both empty). Then, when $\mu_1 = \mu_2 = 0$, a reduction in λ_1 below λ_2 keeps (λ_1, μ_1) in region R_{1MH}, in which bidding in the FPA does not depend on (λ_1, μ_1). Hence $R^S > R^F$ for any $\lambda_1 < \lambda_2$ because when the distribution of v_1 becomes stronger, R^S increases but R^F remains constant (as type 1_H puts a probability mass on the bid v_L when $\lambda_1 < \lambda_2$). This feature of the FPA when the support is binary is responsible for the difference between the two settings.

4.2.2 Some specific asymmetries

The results in Subsection 4.2.1 allow to determine the effects of a few particular asymmetries.

First order stochastic dominance, Mean preserving spreads Starting from a symmetric setting with $(\lambda_1, \mu_1, \eta_1) = (\lambda_2, \mu_2, \eta_2)$, consider a change in the distribution of v_1 which satisfies (1). Then Proposition 3 reveals that $R^S > R^F$ if $\lambda_1 \geq \lambda_2$, or if $\lambda_1 < \lambda_2$ and (1) holds with equality, and these conditions allow to conclude that $R^S > R^F$ in the following cases:

- (i) $(\lambda_1, \mu_1) = (\lambda_2, \mu_2 - \varepsilon)$ for some $\varepsilon > 0$ – then (λ_1, μ_1) belongs to the segment connecting $(\lambda_2, 0)$ to (λ_2, μ_2) in Figure 1;
- (ii) $(\lambda_1, \mu_1) = (\lambda_2 - \varepsilon, \mu_2 + \varepsilon)$ for some $\varepsilon > 0$ – then (λ_1, μ_1) belongs to the segment connecting $(0, \lambda_2 + \mu_2)$ to (λ_2, μ_2) in Figure 1;
- (iii) the distribution of v_1 is a mean preserving spread of the distribution of v_2, that is $\lambda_1 = \lambda_2 + \frac{\varepsilon}{2}$, $\mu_1 = \mu_2 - \varepsilon$, $\eta_1 = \eta_2 + \frac{\varepsilon}{2}$ for some $\varepsilon > 0$ – then (λ_1, μ_1) belongs to the triangle in Figure 1 with vertices $(\lambda_2, 0)$, (λ_2, μ_2), $(\lambda_2 + \mu_2, 0)$.[36]

Proposition 3 does not cover cases such that $\lambda_1 < \lambda_2$ and (1) holds strictly – then the distribution of v_1 first order stochastically dominates the distribution of v_2. But the ranking between R^F and R^S is not clear cut, as if for instance $\mu_1 = \mu_2$ then $(\lambda_1, \mu_1, \eta_1) = (\lambda_2 - \varepsilon, \mu_2, \eta_2 + \varepsilon)$ for some $\varepsilon > 0$ and $R^S > R^F$ for a small ε by Proposition 3, but $R^F > R^S$ may hold for a large ε, for instance if $(\lambda_1, \mu_1) = (0, 0.33)$, $(\lambda_2, \mu_2) = (0.65, 0.33)$. The reason is that when $(\lambda_1, \mu_1) \in R_{1M}$, R^F is decreasing and convex in λ_1, hence as λ_1 decreases, R^F increases at an increasing rate (i.e., the rate of increase is greater the greater is ε), whereas R^S is decreasing linearly in λ_1. As a result, in some cases a large decrease in λ_1 (i.e., a large ε) implies $R^F > R^S$.

[35] We consider $\mu_1 = \mu_2 < \lambda_1 = \lambda_2$ to fix the ideas, but a similar argument would apply if $\mu_1 = \mu_2$ were greater than $\lambda_1 = \lambda_2$.

[36] More in general, $R^S > R^F$ holds if the distribution of v_2 dominates the distribution of v_1 in the sense of second order stochastic dominance. Because of (1), the distribution of v_2 cannot be a mean preserving spread of the distribution of v_1.

When $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$, that is when both the probability of value v_L and the probability of v_M are lower for bidder 1 than the analogous probabilities for bidder 2, Proposition 4 provides some results. If (λ_2, μ_2) satisfies $\lambda_2 \leq \mu_2$ and (19), then $R^F > R^S$ holds if $(\lambda_1, \mu_1) = (0, 0)$. That is, a strong enough improvement in the distribution of v_1 leads to $R^F > R^S$. If instead $\lambda_2 > \mu_2$ and (20) is satisfied, then $R^F > R^S$ if $(\lambda_1, \mu_1) = (\lambda_2 - \mu_2, 0)$, but not necessarily if $(\lambda_1, \mu_1) = (0, 0)$, as reducing λ_1 below $\lambda_2 - \mu_2$ (when $\mu_1 = 0$) does not change R^F but increases R^S.

Maskin and Riley (2000a) consider a few particular classes of asymmetries, one of which is called shift, another is called stretch. In a setting with continuously distributed values, Maskin and Riley (2000a) prove that the FPA produces a higher revenue than the SPA for any shift and any stretch, under suitable assumptions on the initial distribution which is then shifted or stretched [Kirkegaard (2012) proves that $R^F > R^S$ under slightly relaxed assumptions]

Shift In our context, the shift asymmetry is obtained by assuming that (λ_2, μ_2) is such that $\lambda_2 + \mu_2 = 1$, that is $\eta_2 = 0$, and $\lambda_1 = 0$, $\mu_1 = \lambda_2$, $\eta_1 = \mu_2$. This means that only the values v_L and v_M are possible for bidder 2, whereas only v_M and v_H are possible for bidder 1, with $\Pr\{v_1 = v_M\} = \Pr\{v_2 = v_L\}$ and $\Pr\{v_1 = v_H\} = \Pr\{v_1 = v_M\}$. Hence, the distribution for v_1 coincides with the distribution for v_2 shifted to the right by Δ. Notice that $\lambda_2 + \mu_2 = 1$ makes (19) satisfied when $\lambda_2 \leq \mu_2$, and (20) satisfied when $\lambda_2 > \mu_2$.

When $\lambda_2 \leq \mu_2$ – that is when $\lambda_2 \leq \frac{1}{2}$, as $\lambda_2 + \mu_2 = 1$ – we know that $R^F > R^S$ if (λ_1, μ_1) is close to $(0, 0)$. Since $\lambda_1 = 0$, $\mu_1 = \lambda_2$, it follows that $R^F > R^S$ if λ_2 is close to zero, and numerical methods shows that $R^F > R^S$ if and only if $\lambda_2 < 0.3182$. Therefore when $\lambda_2 \leq \frac{1}{2}$, a shift makes R^F greater than R^S if and only if the distribution of v_2 is not too weak, because otherwise the distribution of v_1 is not strong enough to imply $R^F > R^S$.

When $\lambda_2 > \mu_2$ – that is when $\lambda_2 > \frac{1}{2}$ – the set of (λ_1, μ_1) such that $R^F > R^S$ is concentrated around $(\lambda_2 - \mu_2, 0)$, a point on the horizontal axis in the space (λ_1, μ_1). But the shift implies $(\lambda_1, \mu_1) = (0, \lambda_2)$, which lies on the vertical axis, and numerical methods show that it does not belong to the set such that $R^F > R^S$.[37]

Proposition 5 (Shift) Under the shift asymmetry, $R^F > R^S$ if and only if $\lambda_2 < 0.3182$.

Stretch In our context, the stretch asymmetry may be represented by assuming that (λ_2, μ_2) is such that $\lambda_2 + \mu_2 = 1$, that is $\eta_2 = 0$, and $\lambda_1 = \alpha\lambda_2$, $\mu_1 = \alpha\mu_2$, $\eta_1 = 1 - \alpha$ for some $\alpha \in (0, 1)$. Therefore only the values v_L and v_M are possible for bidder 2, whereas for bidder 1, v_L, v_M, v_H are all possible, but the probabilities of v_L, v_M are a fraction of the probabilities of v_L, v_M for bidder 2 – the rest of the probability is allocated to v_H. As α varies in $(0, 1)$, (λ_1, μ_1) moves along the segment connecting $(0, 0)$ to (λ_2, μ_2).

When $\lambda_2 \leq \mu_2$, if α is close to 0 then it is immediate that (λ_1, μ_1) is close to $(0, 0)$ and $R^F > R^S$. But as α increases, λ_1, μ_1 increase and become close to λ_2, μ_2, which implies $R^F < R^S$.[38] This suggests that $R^F > R^S$ if and only if α is sufficiently close to zero, and indeed next proposition establishes that there exists $\alpha^* \in (0, 1)$ such that $R^F > R^S$ if and only if $\alpha < \alpha^*$.

When $\lambda_2 > \mu_2$, the set of (λ_1, μ_1) such that $R^F > R^S$ consists of points near $(\lambda_2 - \mu_2, 0)$, and it is not immediate whether this set includes points on the segment connecting $(0, 0)$ to (λ_2, μ_2). Next proposition

[37] Doni and Menicucci (2013) consider the same type of shift and use the inequality $U^F \geq U^S$ as a sufficient condition for $R^S > R^F$. In our context, $U^F \geq U^S$ holds if and only if $\lambda_2 \geq \frac{2}{5}$, hence $R^S > R^F$ when $\lambda_2 \geq \frac{2}{5}$. Our Proposition 5 determines the precise set of λ_2 such that $R^S > R^F$, that is the interval $(0.3182, 1)$.

[38] This may look as a consequence of Proposition 3, but in fact Proposition 3 is proved under the assumption that $\lambda_2 + \mu_2 < 1$ which is violated here. However, in the proof of Proposition 6 we show that $R^F < R^S$ when α is close to 1.

establishes that actually the intersection between the set and the segment is non-empty, but becomes tiny when λ_2 is large.

Proposition 6 (Stretch) Under the stretch asymmetry,
(a) if $\lambda_2 \leq \mu_2$, then there exists $\alpha^* \in (0,1)$ such that $R^F > R^S$ if and only if $\alpha \in (0, \alpha^*)$;
(b) if $\lambda_2 > \mu_2$, then there exists α^*, α^{**} in $(0,1)$ such that $\alpha^* < \lambda_2 - \mu_2 < \alpha^{**}$ and $R^F > R^S$ if and only if $\alpha \in (\alpha^*, \alpha^{**})$,[39] but the interval (α^*, α^{**}) has vanishing width (i.e., $\alpha^{**} - \alpha^* \to 0$), when λ_2 tends to 1.

We remark that Propositions 5 and 6 provide a significantly more nuanced picture with respect to the results in Maskin and Riley (2000a) and Kirkegaard (2012), as $R^S > R^F$ emerges in a variety of cases.

5 Conclusions

In this paper we have determined the closed form of the unique equilibrium in the FPA for a two-bidder setting with asymmetric value distributions. Although our analysis is limited in terms of the set of possible valuations for each bidder, our results do not need restrictions on the distributions over the given set and allow a careful comparison between the FPA and the SPA in terms of revenue and in terms of the effects of an ex ante change in one (or both) value distribution on the resulting equilibrium bidding in FPA.

Arozamena and Cantillon (2004) consider a procurement setting in which the type of each bidder coincides with the bidder's cost to produce the object the auctioneer is interested in, and suppose a bidder may make an observable investment, before he learns the own type and before the auction takes place, which improves the own ex ante cost distribution. Arozamena and Cantillon (2004) inquire how the bidder's incentive to invest depends on whether the auction is a FPA or a SPA, assuming that only one bidder can make the investment and imposing some restrictions on the effect of the investment on the cost distribution. Our Proposition 1 can be readily adapted to a procurement setting in which the production cost for each bidder belongs to a set $\{c_L, c_M, c_H\}$ and the cost distributions are asymmetric. Hence it is possible to examine the question addressed by Arozamena and Cantillon (2004) without restrictions on the post-investment distribution, and to study more general investment games in which both bidders can invest, possibly starting from asymmetric situations in order to find out whether an initially advantaged bidder has a greater or smaller incentive to invest than a disadvantaged bidder, while comparing the FPA with the SPA.

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6 Appendix

6.1 Proof of Lemma 1

Proof First notice that u^F_{1H} and u^F_{2H} are both equal to $v_H - \bar{b}_H$, hence

$$u^F_{1H} = u^F_{2H} \tag{21}$$

The proof that $\lambda_1 + \mu_1 < \lambda_2 + \mu_2$ implies $\bar{b}_{1M} < \bar{b}_{2M}$ and the proof that $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$ implies $\bar{b}_{1M} = \bar{b}_{2M}$ relies several times on contradiction arguments which lead to violating (21).

Part 1: Proof that $\lambda_1 + \mu_1 < \lambda_2 + \mu_2$ implies $\bar{b}_{1M} < \bar{b}_{2M}$ We suppose that $\lambda_1 + \mu_1 < \lambda_2 + \mu_2$ and $\bar{b}_{1M} \geq \bar{b}_{2M}$, and show that these inequalities imply that (21) is violated. We distinguish the case of $\bar{b}_{2M} > v_L$ from the case of $\bar{b}_{2M} = v_L$. In order to deal with the second case, we first prove that $\bar{b}_{1M} > v_L$.

Step 1.1: Proof that (21) is violated when $\bar{\mathbf{b}}_{2M} > \mathbf{v}_L$ Consider type 1_H and notice that by bidding $\bar{b}_{2M}$, he wins with probability $\lambda_2 + \mu_2$;[40] hence obtains expected utility $(v_H - \bar{b}_{2M})(\lambda_2 + \mu_2)$ and $u^F_{1H} \geq (v_H - \bar{b}_{2M})(\lambda_2 + \mu_2)$. Now consider type 2_H and notice that $\bar{b}_{2M}$ is a bid in the support of his mixed strategy, hence it yields type 2_H his equilibrium expected utility u^F_{2H}. By bidding $\bar{b}_{2M}$, type 2_H wins with probability no more than $\lambda_1 + \mu_1$;[41] hence $u^F_{2H} \leq (v_H - \bar{b}_{2M})(\lambda_1 + \mu_1)$. Summarizing, $u^F_{1H} \geq (v_H - \bar{b}_{2M})(\lambda_2 + \mu_2) > (v_H - \bar{b}_{2M})(\lambda_1 + \mu_1) \geq u^F_{2H}$, violating (21).

Step 1.2: Proof that $\bar{\mathbf{b}}_{1M} > \mathbf{v}_L$ The proof is by contradiction. If $\bar{b}_{1M} = v_L$, then $\bar{b}_{1M} \geq \bar{b}_{2M}$ implies $\bar{b}_{2M} = v_L$ and types 1_M, 2_M both bid v_L with probability 1. In case that bidder 1 has value v_M and bidder 2 has value v_M, the tie-breking rule makes either bidder win with probability $\frac{1}{2}$, paying v_M. But for type 1_M (for instance) it is profitable to increase his bid from v_L to $v_L + \varepsilon$ with a small $\varepsilon > 0$, as then against 2_M his probability to win increases from $\frac{1}{2}$ to 1, and he pays $v_L + \varepsilon$ when he wins rather than v_M; against types $2_L, 2_H$, his probability to win weakly increases, and his payment upon winning increases by ε. Hence his probability to win increases at least by $\frac{1}{2}\mu_2$, thus his expected utility increases at least by $\frac{1}{2}\mu_2 v_H - \varepsilon$, which is positive if ε is small enough.

Step 1.3: Proof that (21) is violated when $\bar{\mathbf{b}}_{2M} = \mathbf{v}_L$ Consider type 1_H and notice that by bidding $v_L + \varepsilon$ for a small $\varepsilon > 0$, he wins with probability greater than $\lambda_2 + \mu_2$;[42] hence $u^F_{1H} > (2\Delta - \varepsilon)(\lambda_2 + \mu_2)$ and $u^F_{1H} \geq 2\Delta(\lambda_2 + \mu_2)$ since ε can be any number close to 0. Now consider type 2_H and notice that by bidding $v_L + \varepsilon$ for a small $\varepsilon > 0$, he wins with probability less than $\lambda_1 + \mu_1$.[43] Moreover, $v_L + \varepsilon$ is a bid in the support $[v_L, \bar{b}_H]$ of the mixed strategy of type 2_H because $v_L < \bar{b}_{1M} < \bar{b}_H$; hence $u^F_{2H} < (2\Delta - \varepsilon)(\lambda_1 + \mu_1) < 2\Delta(\lambda_1 + \mu_1)$. It follows that $u^F_{1H} > u^F_{2H}$, which violates (21).

Part 2: Proof that $\lambda_2 + \mu_2 = \lambda_1 + \mu_1$ implies $\bar{b}_{1M} = \bar{b}_{2M}$ We suppose that $\lambda_2 + \mu_2 = \lambda_1 + \mu_1$ and $\bar{b}_{1M} > \bar{b}_{2M}$ (or, similarly, $\bar{b}_{1M} < \bar{b}_{2M}$), and show that these inequalities imply that (21) is violated. We distinguish the case of $\bar{b}_{2M} > v_L$ from the case of $\bar{b}_{2M} = v_L$.

Step 2.1: Proof that (21) is violated when $\bar{\mathbf{b}}_{2M} > \mathbf{v}_L$ Since $\bar{b}_{2M} > v_L$, by bidding $\bar{b}_{2M}$ type 1_H wins with probability $\lambda_2 + \mu_2$; hence $u^F_{1H} \geq (v_H - \bar{b}_{2M})(\lambda_2 + \mu_2)$. Conversely, by bidding $\bar{b}_{2M}$ type 2_H wins with

[40] The winning probability is $\lambda_2 + \mu_2$ because the bid $\bar{b}_{2M}$ beats types $2_L, 2_M$ with probability 1 but is defeated by type 2_H (recall that no bidder type puts a probability mass at any bid greater than v_L).

[41] The winning probability is no more than $\lambda_1 + \mu_1$ because the bid $\bar{b}_{2M}$ is not larger than $\bar{b}_{1M}$, hence at most it beats types $1_L, 1_M$ with probability 1, but is defeated by type 1_H.

[42] The winning probability is greater than $\lambda_2 + \mu_2$ because the bid $v_L + \varepsilon$ beats types $2_L, 2_M$ with probability 1, and type 2_H with positive probability as $\underline{b}_{2H} = v_L$.

[43] The winning probability is less than $\lambda_1 + \mu_1$ because $v_L + \varepsilon < \bar{b}_{1M}$ implies that the bid $v_L + \varepsilon$ does not beat type 1_M with probability 1.

probability smaller than $\lambda_1+\mu_1$; hence $u_{2H}^F<(v_H-\bar{b}_{2M})(\lambda_1+\mu_1)$. Therefore $u_{1H}^F\geq(v_H-\bar{b}_{2M})(\lambda_2+\mu_2)=(v_H-\bar{b}_{2M})(\lambda_1+\mu_1)>u_{2H}^F$, contradicting (21).

Step 2.2: Proof that (21) is violated when $\bar{b}_{2M}=v_L$ Consider type 1_H and notice that by bidding $v_L+\varepsilon$ for a small $\varepsilon>0$, he wins with probability greater than $\lambda_2+\mu_2$; hence $u_{1H}^F>(2\Delta-\varepsilon)(\lambda_2+\mu_2)$ and $u_{1H}^F\geq 2\Delta(\lambda_2+\mu_2)$. Now consider type 2_H and notice that by bidding $v_L+\varepsilon$ for a small $\varepsilon>0$, he wins with probability less than $\lambda_1+\mu_1$ as $\bar{b}_{1M}>v_L+\varepsilon$; hence $u_{2H}^F<(2\Delta-\varepsilon)(\lambda_1+\mu_1)<2\Delta(\lambda_1+\mu_1)$. Again, (21) is contradicted.

Proof of $G_1(\bar{b}_{2M})=\lambda_2+\mu_2$ For type 1_H, the bid $\bar{b}_{2M}$ yields expected utility u_{1H}^F since $\bar{b}_{2M}\in[\bar{b}_{1M},\bar{b}_H]$, hence $u_{1H}^F=(v_H-\bar{b}_{2M})(\lambda_2+\mu_2)$ as $\bar{b}_{2M}$ beats types $2_L,2_M$ but not 2_H. For type 2_H, the bid $\bar{b}_{2M}$ yields expected utility u_{2H}^F since $\bar{b}_{2M}\in[\bar{b}_{2M},\bar{b}_H]$, hence $u_{2H}^F=(v_H-\bar{b}_{2M})p$, where p is bidder 2's probability to win by bidding $\bar{b}_{2M}$.[44] From $u_{1H}^F=u_{2H}^F$ we obtain $(v_H-\bar{b}_{2M})(\lambda_2+\mu_2)=(v_H-\bar{b}_{2M})p$, hence $p=\lambda_2+\mu_2$.

6.2 Proof of Proposition 1

In order to derive (9), notice that: (i) as $\bar{b}_{1M}\in(v_L,\bar{b}_{1M}]$, (3) implies $\frac{\rho_1}{\lambda_1+\mu_1}G_2(\bar{b}_{1M})=\rho_2$; (ii) as $\bar{b}_{1M}\in[\bar{b}_{1M},\bar{b}_H]$, (4) implies $(1+\frac{\rho_1}{\lambda_1+\mu_1})G_2(\bar{b}_{1M})=\rho_1+\lambda_2+\mu_2$. Joining (i) and (ii) reveals that $F(\rho_1,\rho_2)=0$ in (9) needs to hold.

Case of $F(\lambda_1,\lambda_2)<0$ When $F(\lambda_1,\lambda_2)<0$, equality (9) is satisfied by $\rho_1=\lambda_1$ and ρ_2 equal to the unique solution to $F(\lambda_1,\rho_2)=0$, that is $\rho_2=\lambda_1\frac{\lambda_1+\lambda_2+\mu_2}{2\lambda_1+\mu_1}$, which belongs to $(\lambda_2,\lambda_2+\mu_2)$ and through (8) it determines $\bar{b}_{1M}$, $\bar{b}_{2M}$, $\bar{b}_H$. Using the expression of R^F at the end of Subsection 3.1.2 yields

$$R_{2M}^F=\lambda_1\rho_2v_L+\int_{v_L}^{\bar{b}_H}bd\left(G_1(b)G_2(b)\right)=\bar{b}_H-\int_{v_L}^{\bar{b}_H}G_1(b)G_2(b)db$$

and G_1,G_2 are obtained from (3)-(6) to obtain[45]

$$G_1(b)=\begin{cases}\frac{\rho_1\Delta}{v_M-b} & \text{for each } b\in(v_L,\bar{b}_{2M}]\\ \frac{v_H-\bar{b}_H}{v_H-b} & \text{for each } b\in[\bar{b}_{2M},\bar{b}_H]\end{cases}\qquad G_2(b)=\begin{cases}\frac{\rho_2\Delta}{v_M-b} & \text{for each } b\in(v_L,\bar{b}_{1M}]\\ \frac{v_H-\bar{b}_H}{v_H-b} & \text{for each } b\in[\bar{b}_{1M},\bar{b}_H]\end{cases}\tag{22}$$

Hence

$$\begin{aligned}R_{2M}^F&=\bar{b}_H-\int_{v_L}^{\bar{b}_{1M}}\frac{\lambda_1\rho_2\Delta^2}{(v_M-b)^2}db-\int_{\bar{b}_{1M}}^{\bar{b}_{2M}}\frac{\lambda_1\Delta(v_H-\bar{b}_H)}{(v_M-b)(v_H-b)}db-\int_{\bar{b}_{2M}}^{\bar{b}_H}\frac{(v_H-\bar{b}_H)^2}{(v_H-b)^2}db\\&=\bar{b}_H-\lambda_1\rho_2\frac{\bar{b}_{1M}-v_L}{v_M-\bar{b}_{1M}}\Delta-\lambda_1(v_H-\bar{b}_H)\ln\left(\frac{(v_H-\bar{b}_{2M})(v_M-\bar{b}_{1M})}{(v_M-\bar{b}_{2M})(v_H-\bar{b}_{1M})}\right)-\frac{(v_H-\bar{b}_H)(\bar{b}_H-\bar{b}_{2M})}{v_H-\bar{b}_{2M}}\\&=v_L+\left(2-\rho_2\mu_1-(2-\lambda_2-\mu_2)(\lambda_1+\lambda_2+\mu_2)-\lambda_1(\lambda_1+\lambda_2+\mu_2)\ln\left(\frac{\lambda_2+\mu_2+\lambda_1}{2\lambda_1+\mu_1}\right)\right)\Delta\end{aligned}\tag{23}$$

[44] Since $\bar{b}_{1M}\leq\bar{b}_{2M}$, it follows that p is at least $\lambda_1+\mu_1$.

[45] When $(\lambda_1,\mu_1)\in R_{2M}$, no profitable deviation exists for any bidder type. Precisely, for type 1_L the equilibrium utility is 0, and his utility is still 0 if he bids less than v_L, whereas it is negative if he bids more than v_L. For type 1_M, a bid $b\in[\bar{b}_{1M},\bar{b}_H]$ yields utility $(v_M-b)G_2(b)=(v_H-b-\Delta)G_2(b)=u_{1H}^F-\Delta G_2(b)$, in which the second equality follows from (4). At $b=\bar{b}_{1M}$, $u_{1H}^F-\Delta G_2(b)$ coincides with u_{1M}^F because of (3), and for $b\in(\bar{b}_{1M},\bar{b}_H]$, $u_{1H}^F-\Delta G_2(b)$ decreases, hence it is smaller than u_{1M}^F. For type 1_H, a bid $b\in[v_L,\bar{b}_{1M}]$ yields utility $(v_H-b)G_2(b)=(v_M-b+\Delta)G_2(b)=u_{1M}^F+\Delta G_2(b)$, in which the second equality follows from (3). At $b=\bar{b}_{1M}$, $u_{1M}^F+\Delta G_2(b)$ coincides with u_{1H}^F because of (4), and for $b\in[v_L,\bar{b}_{1M})$, $u_{1M}^F+\Delta G_2(b)$ increases, hence it is smaller than u_{1H}^F. Similar arguments apply to types $2_L,2_M,2_H$, and when $(\lambda_1,\mu_1)\in R_{1M}\cup R_{1MH}$.

Case of $F(\lambda_1, \lambda_2) \geq 0 > F(\lambda_1 + \mu_1, \lambda_2)$ In this case, (9) is satisfied by $\rho_2 = \lambda_2$ and ρ_1 equal to the unique solution to $F(\rho_1, \lambda_2) = 0$, that is $\rho_1 = \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2(\lambda_1 + \mu_1)} - \frac{1}{2}\mu_2$, which belongs to $[\lambda_1, \lambda_1 + \mu_1)$,[46] and through (8) it determines $\bar{b}_{1M}$, $\bar{b}_{2M}$, $\bar{b}_H$. Hence the expected revenue is

$$\begin{aligned} R_{1M}^F &= \rho_1\lambda_2 v_L + \int_{v_L}^{\bar{b}_H} bd\left(G_1(b)G_2(b)\right) = \bar{b}_H - \int_{v_L}^{\bar{b}_H} G_1(b)G_2(b)db \\ &= \bar{b}_H - \int_{v_L}^{\bar{b}_{1M}} \frac{\rho_1\lambda_2\Delta^2}{(v_M - b)^2}db - \int_{\bar{b}_{1M}}^{\bar{b}_{2M}} \frac{\rho_1(v_H - \bar{b}_H)}{(v_M - b)(v_H - b)}\Delta db - \int_{\bar{b}_{2M}}^{\bar{b}_H} \frac{(v_H - \bar{b}_H)^2}{(v_H - b)^2}db \\ &= \bar{b}_H - \frac{\rho_1\lambda_2(\bar{b}_{1M} - v_L)\Delta}{v_M - \bar{b}_{1M}} - \rho_1(v_H - \bar{b}_H)\ln\left(\frac{(v_H - \bar{b}_{2M})(v_M - \bar{b}_{1M})}{(v_M - \bar{b}_{2M})(v_H - \bar{b}_{1M})}\right) - \frac{(v_H - \bar{b}_H)(\bar{b}_H - \bar{b}_{2M})}{v_H - \bar{b}_{2M}} \end{aligned} \quad (24)$$

Case of $F(\lambda_1 + \mu_1, \lambda_2) \geq 0$ In this case, no $\rho_1 < \lambda_1 + \mu_1$ satisfies $F(\rho_1, \lambda_2) = 0$. Therefore $\rho_1 \geq \lambda_1 + \mu_1$, that is type 1_M bids v_L with probability 1 – hence $\bar{b}_{1M} = v_L$ – and if $\rho_1 > \lambda_1 + \mu_1$ then also type 1_H bids v_L with positive probability. The utility of type 1_H from bidding v_L is $2\lambda_2\Delta$, hence $\bar{b}_H = v_H - 2\lambda_2\Delta$, and also the equilibrium utility of type 2_H is $2\lambda_2\Delta$. It is still the case that $\bar{b}_{2M}$ is given by (8), and type 2_H's utility from bidding $\bar{b}_{2M}$ is $\rho_1 + \lambda_2 + \mu_2$. Hence $2\lambda_2\Delta$ needs to be equal to $\rho_1 + \lambda_2 + \mu_2$ and it follows that $\rho_1 = \lambda_2 - \mu_2$, which is indeed greater than $\lambda_1 + \mu_1$.

The expected revenue has the same expression as (24), but $\bar{b}_{1M}$ is replaced by v_L:

$$\begin{aligned} R_{1MH}^F &= \bar{b}_H - \rho_1(v_H - \bar{b}_H)\ln\left(\frac{v_H - \bar{b}_{2M}}{2(v_M - \bar{b}_{2M})}\right) - \frac{(v_H - \bar{b}_H)(\bar{b}_H - \bar{b}_{2M})}{v_H - \bar{b}_{2M}} \\ &= v_L + \left(2 - 2\lambda_2(2 - \lambda_2 - \mu_2) - 2(\lambda_2 - \mu_2)\lambda_2\ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right)\right)\Delta \end{aligned} \quad (25)$$

Bidders' rents The bidders' rents are given in the following table, in which the common factor Δ is omitted. For P_{1M}, ρ_1 is given by the expression in (11):

equilibrium \ bidder type	1_M	1_H	2_M	2_H
P_{2M}	$\lambda_1\frac{\lambda_1+\lambda_2+\mu_2}{2\lambda_1+\mu_1}$	$\lambda_1 + \lambda_2 + \mu_2$	λ_1	$\lambda_1 + \lambda_2 + \mu_2$
P_{1M}	λ_2	$\rho_1 + \lambda_2 + \mu_2$	ρ_1	$\rho_1 + \lambda_2 + \mu_2$
P_{1MH}	λ_2	$2\lambda_2$	$\lambda_2 - \mu_2$	$2\lambda_2$

(26)

6.3 Proof of Proposition 2

(i) When $(\lambda_1, \mu_1) \in R_{1MH}$, we have that $G_{1M}(v_L) = 1$, hence the inequality in (i) is satisfied. When $(\lambda_1, \mu_1) \in R_{1M} \cup R_{2M}$, we have that $G_{1M}(b) = \frac{1}{\mu_1}\left(\frac{\rho_1\Delta}{v_M - b} - \lambda_1\right)$, $G_{2M}(b) = \frac{1}{\mu_2}\left(\frac{\rho_2\Delta}{v_M - b} - \lambda_2\right)$ and $G_{2M}(b) < G_{1M}(b)$ reduces to $(\lambda_1\mu_2 - \lambda_2\mu_1)(v_M - b) < (\mu_2\rho_1 - \mu_1\rho_2)\Delta$. If $\lambda_1\mu_2 - \lambda_2\mu_1 > 0$, then the inequality is more restrictive when b is close to v_L, hence it holds if and only if $\mu_1(\rho_2 - \lambda_2) \leq \mu_2(\rho_1 - \lambda_1)$, which is satisfied if $(\lambda_1, \mu_1) \in R_{1M}$ as $\rho_1 \geq \lambda_1$, $\rho_2 = \lambda_2$, but is violated if $(\lambda_1, \mu_1) \in R_{2M}$ as $\rho_1 = \lambda_1$, $\rho_2 > \lambda_2$. If instead $\lambda_1\mu_2 - \lambda_2\mu_1 \leq 0$, then the inequality is more restrictive at $b = \bar{b}_{1M}$, but it is definitely satisfied at $b = \bar{b}_{1M}$ as it is equivalent to $G_{2M}(\bar{b}_{1M}) < G_{1M}(\bar{b}_{1M})$, and $G_{2M}(\bar{b}_{1M}) < 1$ as $\bar{b}_{1M} < \bar{b}_{2M}$ by Lemma 1, $G_{1M}(\bar{b}_{1M}) = 1$.

(ii) We have that $G_{1H}(b) = \frac{G_1(b) - \lambda_1 - \mu_1}{1 - \lambda_1 - \mu_1}$, $G_{2H}(b) = \frac{G_2(b) - \lambda_2 - \mu_2}{1 - \lambda_2 - \mu_2}$. Since $G_1(b) = G_2(b)$ for each $b \in [\bar{b}_{2M}, \bar{b}_H)$, it follows that the strict version of (1) implies $G_{2H}(b) < G_{1H}(b)$ for each $b \in [\bar{b}_{2M}, \bar{b}_H)$.

[46] Precisely, $\rho_1 = \lambda_1$ when $F(\lambda_1, \lambda_2) = 0$, ρ_1 belongs to $(\lambda_1, \lambda_1 + \mu_1)$ if $F(\lambda_1, \lambda_2) > 0$.

(iii) When $(\lambda_1, \mu_1) \in R_{1MH}$, the inequality $G_1(b) \leq G_2(b)$ for $b \in [v_L, \bar{b}_{2M}]$ reduces to $\frac{\lambda_2 - \mu_2}{v_M - b} \leq \frac{2\lambda_2}{v_H - b}$, or to $(\lambda_2 - \mu_2)\Delta \leq (\lambda_2 + \mu_2)(v_M - b)$, and at $b = \bar{b}_{2M}$ this inequality holds with equality. For $b \in [\bar{b}_{2M}, \bar{b}_H]$, the inequality $G_1(b) \leq G_2(b)$ holds with equality.

When $(\lambda_1, \mu_1) \in R_{1M} \cup R_{2M}$, the inequality $G_1(b) \leq G_2(b)$ for $b \in [v_L, \bar{b}_{1M}]$ is $\frac{\rho_1}{v_M - b} \leq \frac{\rho_2}{v_M - b}$, which holds since $\rho_1 \leq \rho_2$ for each $(\lambda_1, \mu_1) \in R_{1M} \cup R_{2M}$. For $b \in [\bar{b}_{1M}, \bar{b}_{2M}]$, the inequality $G_1(b) \leq G_2(b)$ is $\frac{\rho_1}{v_M - b} \leq \frac{\rho_1 + \lambda_2 + \mu_2}{v_H - b}$, or $\rho_1\Delta \leq (\lambda_2 + \mu_2)(v_M - b)$, and at $b = \bar{b}_{2M}$ this holds with equality. For $b \in [\bar{b}_{2M}, \bar{b}_H]$, $G_1(b) \leq G_2(b)$ holds with equality.

(iv) We prove that $G_1(b) \leq G_2^{\text{sym}}(b)$ when $\lambda_1 \leq \lambda_2$. Then $\rho_1 \leq \lambda_2$ and $\bar{\beta}_{2M} \leq \bar{b}_{2M}$, $\bar{\beta}_{2H} \leq \bar{b}_H$. For $b \in [v_L, \bar{\beta}_{2M}]$, the inequality $G_1(b) \leq G_2^{\text{sym}}(b)$ is $\frac{\rho_1}{v_M - b} \leq \frac{\lambda_2}{v_M - b}$, which holds as $\rho_1 \leq \lambda_2$. For $b \in [\bar{\beta}_{2M}, \bar{b}_{2M}]$, the inequality $G_1(b) \leq G_2^{\text{sym}}(b)$ is $\frac{\rho_1}{v_M - b}\Delta \leq \frac{v_H - \bar{\beta}_{2H}}{v_H - b}$, or $\rho_1\Delta \leq (2\lambda_2 + \mu_2 - \rho_1)(v_M - b)$, and at $b = \bar{b}_{2M}$ it reduces to $\rho_1 \leq (2\lambda_2 + \mu_2 - \rho_1)\frac{\rho_1}{\lambda_2 + \mu_2}$, which is satisfied as $\rho_1 \leq \lambda_2$. For $b \in [\bar{b}_{2M}, \bar{\beta}_{2H}]$, the inequality $G_1(b) \leq G_2^{\text{sym}}(b)$ is $\frac{v_H - \bar{b}_H}{v_H - b} \leq \frac{v_H - \bar{\beta}_{2H}}{v_H - b}$, which holds since $\bar{\beta}_{2H} \leq \bar{b}_H$.[47]

Now we show that $G_2^{\text{sym}}(b) \leq G_1(b)$ when $\lambda_1 > \lambda_2$. Then $(\lambda_1, \mu_1) \in R_{2M}$ and $\rho_1 = \lambda_1 > \lambda_2$, $\bar{b}_{2M} < \bar{\beta}_{2M}$, $\bar{b}_H < \bar{\beta}_{2H}$. For $b \in [v_L, \bar{b}_{2M}]$, the inequality $G_2^{\text{sym}}(b) \leq G_1(b)$ is $\frac{\lambda_2}{v_M - b} \leq \frac{\rho_1}{v_M - b}$, which holds as $\rho_1 > \lambda_2$. For $b \in [\bar{b}_{2M}, \bar{\beta}_{2M}]$, the inequality $G_2^{\text{sym}}(b) \leq G_1(b)$ reduces to $\frac{\lambda_2}{v_M - b} \leq \frac{\lambda_1 + \lambda_2 + \mu_2}{v_H - b}$, or to $\lambda_2\Delta \leq (\lambda_1 + \mu_2)(v_M - b)$, and at $b = \bar{\beta}_{2M}$ it holds as $\lambda_1 > \lambda_2$. For $b \in [\bar{\beta}_{2M}, \bar{b}_H]$, the inequality $G_2^{\text{sym}}(b) \leq G_1(b)$ is $\frac{v_H - \bar{\beta}_{2H}}{v_H - b} \leq \frac{v_H - \bar{b}_H}{v_H - b}$, which holds since $\bar{b}_H \leq \bar{\beta}_{2H}$.[48]

(v) First notice that $\bar{b}_{1M} \leq \bar{\beta}_{1M}$ and $\bar{b}_H < \bar{\beta}_{1H}$ because $\rho_1 \geq \lambda_1$.

When $(\lambda_1, \mu_1) \in R_{1MH}$, the inequality $G_1^{\text{sym}}(b) \leq G_2(b)$ for $b \in [v_L, \bar{\beta}_{1M}]$ reduces to $\frac{\lambda_1}{v_M - b} \leq \frac{\rho_1 + \lambda_2 + \mu_2}{v_H - b}$, or to $\lambda_1\Delta \leq (\rho_1 + \lambda_2 + \mu_2 - \lambda_1)(v_M - b)$, and at $b = \bar{\beta}_{1M}$ this boils down to $\lambda_1 + \mu_1 \leq \rho_1 + \lambda_2 + \mu_2 - \lambda_1$, which holds because of (1) and $\rho_1 \geq \lambda_1$. For $b \in [\bar{\beta}_{1M}, \bar{b}_H]$, the inequality $G_1^{\text{sym}}(b) \leq G_2(b)$ is $\frac{v_H - \bar{\beta}_{1H}}{v_H - b} \leq \frac{v_H - \bar{b}_H}{v_H - b}$, which holds since $\bar{b}_H < \bar{\beta}_{1H}$.[49]

When $(\lambda_1, \mu_1) \in R_{1M} \cup R_{2M}$, for $b \in [v_L, \bar{b}_{1M}]$ the inequality $G_1^{\text{sym}}(b) \leq G_2(b)$ is $\frac{\lambda_1}{v_M - b} \leq \frac{\rho_2}{v_M - b}$, which holds as $\rho_2 \geq \lambda_1$. For $b \in [\bar{b}_{1M}, \bar{\beta}_{1M}]$, the inequality $G_1^{\text{sym}}(b) \leq G_2(b)$ is proved as above for the case of $b \in [v_L, \bar{\beta}_{1M}]$. For $b \in [\bar{\beta}_{1M}, \bar{b}_H]$, the inequality $G_1^{\text{sym}}(b) \leq G_2(b)$ is proved as above for the case of $b \in [\bar{\beta}_{1M}, \bar{b}_H]$.

(vi). First notice that $\bar{b}_H \leq \bar{\beta}_{1H}$ since $\rho_1 \geq \lambda_1$, but either $\bar{b}_{2M} < \bar{\beta}_{1M}$ or $\bar{\beta}_{1M} < \bar{b}_{2M}$ may hold. For $b \in [v_L, \min\{\bar{b}_{2M}, \bar{\beta}_{1M}\}]$, the inequality $G_1^{\text{sym}}(b) \leq G_1(b)$ reduces to $\frac{\lambda_1}{v_M - b} \leq \frac{\rho_1}{v_M - b}$, which is satisfied since $\rho_1 \geq \lambda_1$. If $\bar{b}_{2M} < \bar{\beta}_{1M}$, then for $b \in [\bar{b}_{2M}, \bar{\beta}_{1M}]$ the inequality $G_1^{\text{sym}}(b) \leq G_1(b)$ reduces to $\frac{\lambda_1}{v_M - b} \leq \frac{\rho_1 + \lambda_2 + \mu_2}{v_H - b}$, or to $\lambda_1\Delta \leq (\rho_1 + \lambda_2 + \mu_2 - \lambda_1)(v_M - b)$, and at $b = \bar{\beta}_{1M}$ this holds because of (1) and $\rho_1 \geq \lambda_1$. If $\bar{\beta}_{1M} < \bar{b}_{2M}$, then for $b \in [\bar{\beta}_{1M}, \bar{b}_{2M}]$ the inequality $G_1^{\text{sym}}(b) \leq G_1(b)$ reduces to $\frac{v_H - \bar{\beta}_{1H}}{v_H - b} \leq \frac{\rho_1\Delta}{v_M - b}$, or to $(2\lambda_1 + \mu_1 - \rho_1)(v_M - b) \leq \rho_1\Delta$. In case that $2\lambda_1 + \mu_1 - \rho_1 \geq 0$, the inequality holds at $b = \bar{\beta}_{1M}$ since $\rho_1 \geq \lambda_1$. In case that $2\lambda_1 + \mu_1 - \rho_1 < 0$, the inequality holds because the left hand side is negative, the right hand side is positive.

(vii) When $(\lambda_1, \mu_1) \in R_{1MH}$, we have that $\bar{b}_{1M} = v_L \leq \bar{\beta}_{2M} = v_M - \frac{\lambda_2}{\lambda_2 + \mu_2}\Delta < \bar{\beta}_{2H} = v_H - (2\lambda_2 + \mu_2)\Delta \leq \bar{b}_H = v_H - 2\lambda_2\Delta$. For $b \in [v_L, \bar{\beta}_{2M}]$, the inequality $G_2(b) \leq G_2^{\text{sym}}(b)$ reduces to $\frac{2\lambda_2}{v_H - b} \leq \frac{\lambda_2}{v_M - b}$, or to $v_M - b \leq \Delta$, which holds with equality at $b = v_L$. For $b \in [\bar{\beta}_{2M}, \bar{\beta}_{2H}]$, the inequality $G_2(b) \leq G_2^{\text{sym}}(b)$ is $\frac{v_H - \bar{b}_H}{v_H - b} \leq \frac{v_H - \bar{\beta}_{2H}}{v_H - b}$, which holds as $\bar{\beta}_{2H} \leq \bar{b}_H$.

When $(\lambda_1, \mu_1) \in R_{1M}$, we have that $\bar{b}_{1M} = v_M - \frac{\rho_1}{\lambda_1 + \mu_1}\Delta \leq \bar{\beta}_{2M} = v_M - \frac{\lambda_2}{\lambda_2 + \mu_2}\Delta$ holds since it is

[47] In fact, for some parameters the inequality $\bar{\beta}_{2H} < \bar{b}_M$ holds, but then $G_1(b) \leq G_2^{sym}(b)$ still holds for each $b \in [v_L, \bar{\beta}_{2H}]$.

[48] In fact, for some parameters the inequality $\bar{b}_H < \bar{\beta}_{2M}$ holds, but then $G_2^{sym}(b) \leq G_1(b)$ still holds for each $b \in [v_L, \bar{b}_{2H}]$.

[49] For some parameters the inequality $\bar{b}_H < \bar{\beta}_{1M}$ holds, but then $G_1^{sym}(b) \leq G_2(b)$ still holds for each $b \in [v_L, \bar{b}_H]$.

equivalent to $\lambda_2(\lambda_1+\mu_1) \le \rho_1(\lambda_2+\mu_2)$, or to $\lambda_2(\lambda_1+\mu_1)+\frac{1}{2}\mu_2(\lambda_2+\mu_2) \le (\lambda_2+\mu_2)\sqrt{\frac{1}{4}\mu_2^2+\lambda_2(\lambda_1+\mu_1)}$, which is equivalent to (1). Moreover, $\bar{\beta}_{2H} = v_H - (2\lambda_2+\mu_2)\Delta \le \bar{b}_H = v_H - (\rho_1+\lambda_2+\mu_2)\Delta$ holds since $\rho_1 \le \lambda_2$. For $b \in [v_L, \bar{b}_{1M}]$, the inequality $G_2(b) \le G_2^{\text{sym}}(b)$ reduces to $\frac{\rho_2}{v_M-b} \le \frac{\lambda_2}{v_M-b}$, which holds since $\rho_2 = \lambda_2$ in region R_{1M}. For $b \in [\bar{b}_{1M}, \bar{\beta}_{2M}]$, the inequality $G_2(b) \le G_2^{\text{sym}}(b)$ is $\frac{v_H-\bar{b}_H}{v_H-b} \le \frac{\lambda_2\Delta}{v_M-b}$, or $(\rho_1+\mu_2)(v_M-b) \le \lambda_2\Delta$, and at $b=\bar{b}_{1M}$ it holds with equality since $\rho_1 = \sqrt{\frac{1}{4}\mu_2^2+\lambda_2(\lambda_1+\mu_1)} - \frac{1}{2}\mu_2$. For $b \in [\bar{\beta}_{2M}, \bar{\beta}_{2H}]$, the inequality $G_2(b) \le G_2^{\text{sym}}(b)$ is $\frac{v_H-\bar{b}_H}{v_H-b} \le \frac{v_H-\bar{\beta}_{2H}}{v_H-b}$, which is satisfied since $\bar{\beta}_{2H} \le \bar{b}_H$.

Now suppose that $\lambda_1 \ge \lambda_2$, hence $(\lambda_1,\mu_1) \in R_{2M}$ and $\bar{b}_{1M} = v_M - \frac{\lambda_1}{\lambda_1+\mu_1}\Delta \le \bar{\beta}_{2M} = v_M - \frac{\lambda_2}{\lambda_2+\mu_2}\Delta$ since $(\lambda_1,\mu_1) \in R_{2M}$.[50] Moreover, $\bar{b}_H = v_H - (\lambda_1+\lambda_2+\mu_2)\Delta < \bar{\beta}_{2H} = v_H - (2\lambda_2+\mu_2)\Delta$ since $\lambda_1 \ge \lambda_2$. For $b \in [v_L, \bar{b}_{1M}]$, the inequality $G_2^{\text{sym}}(b) \le G_2(b)$ is $\frac{\lambda_2}{v_M-b} \le \frac{\rho_2}{v_M-b}$, which holds since $\rho_2 \ge \lambda_2$. For $b \in [\bar{b}_{1M}, \bar{\beta}_{2M}]$, the inequality $G_2^{\text{sym}}(b) \le G_2(b)$ reduces to $\frac{\lambda_2\Delta}{v_M-b} \le \frac{v_H-\bar{b}_H}{v_H-b}$, or to $\lambda_2\Delta \le (\lambda_1+\mu_2)(v_M-b)$, and at $b=\bar{\beta}_{2M}$ it holds since $\lambda_1 \ge \lambda_2$. For $b \in [\bar{\beta}_{2M}, \bar{b}_H]$, the inequality $G_2^{\text{sym}}(b) \le G_2(b)$ is $\frac{v_H-\bar{\beta}_{2H}}{v_H-b} \le \frac{v_H-\bar{b}_H}{v_H-b}$, which is satisfied since $\bar{b}_H \le \bar{\beta}_{2H}$.

6.4 Proof of Lemma 2

The proofs of $u_{1M}^F \ge u_{1M}^S$, $u_{2M}^F \ge u_{2M}^S$, $u_{2H}^F \ge u_{2H}^S$ are in the text, in the first two paragraphs of Subsection 4.1.
For the comparison between $u_{1H}^S = (2\lambda_2+\mu_2)\Delta$ and u_{1H}^F, consider first $(\lambda_1,\mu_1) \in R_{2M}$. Then $u_{1H}^F = (\lambda_1+\lambda_2+\mu_2)\Delta$, hence $u_{1H}^F > u_{1H}^S$ if and only if $\lambda_1 > \lambda_2$. When $(\lambda_1,\mu_1) \in R_{1M}$, we have that $\lambda_1 < \lambda_2$ and $u_{1H}^F = (\lambda_2+\frac{1}{2}\mu_2+\sqrt{\frac{1}{4}\mu_2^2+\lambda_2(\lambda_1+\mu_1)})\Delta$, which is smaller than u_{1H}^S (equal to u_{1H}^S) if (1) holds strictly (if (1) holds with equality). Finally, $u_{1H}^F = 2\lambda_2\Delta$ when $(\lambda_1,\mu_1) \in R_{1MH}$; hence it is immediate that $u_{1H}^F < u_{1H}^S$.

6.5 Proof of Proposition 3

6.5.1 Proof of part (i)

The proof that $\lambda_1 \ge \lambda_2$ implies $U^F > U^S$ is in the text, just before Proposition 3.

6.5.2 Proof of part (ii)

Given a small $\varepsilon > 0$, let R_{1M}^ε consist of $(\lambda_1,\mu_1) \in R_{1M}$ such that $\mu_1 \ge \lambda_2+\mu_2-\lambda_1-\varepsilon$, and let R_{2M}^ε consist of $(\lambda_1,\mu_1) \in R_{2M}$ such that $\mu_1 \ge \mu_2 - \varepsilon$. We now prove that $U^F > U^S$ if $(\lambda_1,\mu_1) \in R_{1M}^\varepsilon \cup R_{2M}^\varepsilon$.

We recall from Lemma 2 that $u_{1M}^F \ge u_{1M}^S$, $u_{2M}^F \ge u_{2M}^S$, hence

$$U^F - U^S \ge \eta_1(u_{1H}^F - u_{1H}^S) + \eta_2(u_{2H}^F - u_{2H}^S) \tag{27}$$

and we focus on the right hand side of (27), that is on the comparison between u_{1H}^F and u_{1H}^S and between u_{2H}^F and u_{2H}^S; recall that $u_{1H}^F - u_{1H}^S \le 0$ and $u_{2H}^F - u_{2H}^S > 0$ when $\lambda_1 < \lambda_2$. We define $U_H^F = \eta_1 u_{1H}^F + \eta_2 u_{2H}^F$, $U_H^S = \eta_1 u_{1H}^S + \eta_2 u_{2H}^S$. Hence the right hand side in (27) is equal to $U_H^F - U_H^S$ and we prove $U^F > U^S$ by showing $U_H^F > U_H^S$.

Step 1 $U^F > U^S$ if $(\lambda_1,\mu_1) \in R_{1M}^\varepsilon$ with ε close to zero.
Proof of Step 1 Consider (λ_1,μ_1) such that $\lambda_1 < \lambda_2$ and $\lambda_1+\mu_1 = \lambda_2+\mu_2$; thus $(\lambda_1,\mu_1) \in R_{1M}$, $\rho_1 = \lambda_2$ and from (26) and (16) it follows that $U_H^F - U_H^S = (1-\lambda_2-\mu_2)(\lambda_2-\lambda_1) > 0$. We use $r = \sqrt{\frac{1}{4}\mu_2^2+\lambda_2(\lambda_1+\mu_1)}$ to write $\frac{\partial(U_H^F-U_H^S)}{\partial\mu_1}$ as $(-1)(\lambda_2+\frac{1}{2}\mu_2+r-2\lambda_2-\mu_2)+(1-\lambda_1-\mu_1)\frac{\lambda_2}{2r}+(1-\lambda_2-\mu_2)(\frac{\lambda_2}{2r}-1)$, which at

[50]For some parameters the inequality $\bar{b}_H < \bar{\beta}_{2M}$ holds, but then $G_2^{sym}(b) \le G_2(b)$ still holds for each $b \in [v_L, \bar{b}_H]$.

$\mu_1 = \lambda_2 + \mu_2 - \lambda_1$ reduces to $-\mu_2 \frac{1-\lambda_2-\mu_2}{2\lambda_2+\mu_2} < 0$ since $r = \lambda_2 + \frac{1}{2}\mu_2$. Moreover, by continuity the derivative is still negative if μ_1 is slightly smaller than $\lambda_2 + \mu_2 - \lambda_1$, that is if $(\lambda_1, \mu_1) \in R^{\varepsilon}_{1M}$ with ε is close to zero. Therefore a small reduction in μ_1 below $\lambda_2 + \mu_2 - \lambda_1$ increases $U^F_H - U^S_H$, which is already positive when $\mu_1 = \lambda_2 + \mu_2 - \lambda_1$. Hence there exists a small $\varepsilon > 0$ such that $U^F_H - U^S_H > 0$ for each $(\lambda_1, \mu_1) \in R^{\varepsilon}_{1M}$.

Step 2 $U^F > U^S$ if $(\lambda_1, \mu_1) \in R^{\varepsilon}_{2M}$ with ε close to zero.

Proof of Step 2 We consider $(\lambda_1, \mu_1) \in R^{\varepsilon}_{2M}$ with $\lambda_1 \leq \lambda_2$ because of part (i). Then $U^F_H - U^S_H = (1-\lambda_1-\mu_1)(\lambda_1-\lambda_2) + (1-\lambda_2-\mu_2)(\lambda_2+\mu_2-\lambda_1-\mu_1)$, which is strictly increasing in λ_1 because of (1) and $\lambda_1 < \lambda_2$. Therefore in order to minimize $U^F_H - U^S_H$, given $\mu_1 \in [\mu_2 - \varepsilon, \mu_2)$, we pick the smallest λ_1 such that $(\lambda_1, \mu_1) \in R^{\varepsilon}_{2M}$; this identifies a point on curve C which satisfies equation $\mu_1 = \frac{1}{\lambda_2}\lambda_1^2 + \frac{\mu_2-\lambda_2}{\lambda_2}\lambda_1$ (from (13)) and $\lambda_1 \in [\lambda_1^{\varepsilon}, \lambda_2]$, where $\lambda_1^{\varepsilon} = \frac{1}{2}\lambda_2 - \frac{1}{2}\mu_2 + \frac{1}{2}\sqrt{(\lambda_2+\mu_2)^2 - 4\varepsilon\lambda_2}$ is the unique λ_1 such that $\frac{1}{\lambda_2}\lambda_1^2 + \frac{\mu_2-\lambda_2}{\lambda_2}\lambda_1 = \mu_2 - \varepsilon$. Given $\mu_1 = \frac{1}{\lambda_2}\lambda_1^2 + \frac{\mu_2-\lambda_2}{\lambda_2}\lambda_1$, we obtain

$$U^F_H - U^S_H = \frac{\lambda_2 - \lambda_1}{\lambda_2}\left(\lambda_1^2 + (1-\lambda_2)\lambda_1 + \mu_2 - (\lambda_2+\mu_2)^2\right) \tag{28}$$

for $\lambda_1 \in [\lambda_1^{\varepsilon}, \lambda_2]$, and $\lambda_1^2 + (1-\lambda_2)\lambda_1 + \mu_2 - (\lambda_2+\mu_2)^2$ is increasing in λ_1, with value $(1-\lambda_2-\mu_2)(\lambda_2+\mu_2) > 0$ at $\lambda_1 = \lambda_2$. By continuity, it has positive value at $\lambda_1 = \lambda_1^{\varepsilon}$ if ε is close to zero. Hence $U^F_H - U^S_H > 0$ for each $(\lambda_1, \mu_1) \in R^{\varepsilon}_{2M}$. ■

6.6 Proof of the C-property

First we describe how R^F depends on λ_1, μ_1. The first step is to recall that R^F is the expectation of the highest submitted bid, with G_1 the c.d.f. of bidder 1's bid, G_2 the c.d.f. of bidder 2's bid. Then from (10)-(12) we obtain next lemma.

Lemma 3(i) When $(\lambda_1, \mu_1) \in R_{1M}$, R^F depends on λ_1, μ_1 only through $\lambda_1 + \mu_1$, and an increase in $\lambda_1 + \mu_1$ reduces R^F.
(ii) When $(\lambda_1, \mu_1) \in R_{2M}$, an increase in λ_1 reduces R^F, an increase in μ_1 increases R^F.
(iii) When $(\lambda_1, \mu_1) \in R_{1MH}$, R^F is constant with respect to (λ_1, μ_1).

When $(\lambda_1, \mu_1) \in R_{1MH}$, the equilibrium in the FPA is P_{1MH} in (12) with $\rho_1 = \lambda_2 - \mu_2$, $\rho_2 = \lambda_2$. Hence G_1, G_2 do not depend on λ_1, μ_1 and this makes R^F constant with respect to (λ_1, μ_1). This feature of P_{1MH} is analogous to a feature of the equilibrium in the setting with binary support described at the end of Subsection 3.1.1 when $\lambda_1 < \lambda_2$.

When $(\lambda_1, \mu_1) \in R_{1M}$, Lemma 3(i) relies on (8) and (11) which reveal that λ_1, μ_1 affect G_1, G_2 only through $\lambda_1 + \mu_1$, and an increase in $\lambda_1 + \mu_1$ increases the probability ρ_1 that bidder 1 bids v_L. This worsens the entire bid distribution of bidder 1 in the sense of first order stochastic dominance. In particular, an increase in ρ_1 lowers $\bar{b}_H$ from (8), and this worsens also the bid distribution of bidder 2. Thus an increase in $\lambda_1 + \mu_1$ worsens G_1 and G_2 and reduces R^F. When $(\lambda_1, \mu_1) \in R_{2M}$, a similar mechanism applies if λ_1 increases because ρ_2 in (10) is strictly increasing in λ_1. Conversely, ρ_2 is strictly decreasing in μ_1, that is an increase in μ_1 reduces ρ_2. This does not affect G_1, but improves G_2 and hence increases R^F. Here a higher μ_1 reduces ρ_2 and thus reduces the utility of type 1_M in (2), which requires more aggressive bidding by type 2_M as determined by a reduction in $G_2(b)$ for $b \in [v_L, \bar{b}_{1M}]$; moreover, $\bar{b}_{1M}$ increases as μ_1 increases.

Proof of the ***C*****-property** Suppose that $(\lambda_1, \mu_1) \in R_{2M}$. Then $R^F - R^S$ is increasing in μ_1 by Lemma 3(ii) and (18). Hence if $R^F - R^S > 0$ for some $(\lambda_1', \mu_1') \in R_{2M}$, then there exists $\mu_1'' > \mu_1'$ such that $(\lambda_1', \mu_1'') \in$ C and $R^F - R^S > 0$.[51] When instead $(\lambda_1, \mu_1) \in R_{1M}$, if $R^F - R^S > 0$ for some $(\lambda_1', \mu_1') \in R_{1M}$

[51] Recall that $\lambda_1' \leq \lambda_2$, as we explained at the beginning of Subsection 4.2.1.

then consider $(\lambda_1'', \mu_1'') \in \mathrm{C}$ such that $\lambda_1' + \mu_1' = \lambda_1'' + \mu_1''$ with $\lambda_1'' > \lambda_1'$. It follows that R^F has the same value when $(\lambda_1, \mu_1) = (\lambda_1', \mu_1')$ as when $(\lambda_1, \mu_1) = (\lambda_1'', \mu_1'')$ because R^F depends on (λ_1, μ_1) only through $\lambda_1 + \mu_1$ by Lemma 3(i), but R^S is smaller when $(\lambda_1, \mu_1) = (\lambda_1'', \mu_1'')$ because of (18) and $\lambda_1'' - \lambda_1' = \mu_1' - \mu_1'' > 0$. Therefore $R^F - R^S$ is greater, hence positive, when $(\lambda_1, \mu_1) = (\lambda_1'', \mu_1'')$. ■

6.7 Proof of Proposition 4

By the virtue of the C-property, if there exists $(\lambda_1, \mu_1) \notin \mathrm{C}$ such that $R^F > R^S$, then there exists at least one $(\lambda_1, \mu_1) \in \mathrm{C}$ such that $R^F > R^S$. Thus we restrict to C and consider the function R^{F-S}, defined for $\lambda_1 \in [\max\{0, \lambda_2 - \mu_2\}, \lambda_2]$. In order to derive the expression of R^{F-S}, we notice that subtracting R^S from R^F_{2M} in (23) we obtain (omitting the factor Δ) $R^F_{2M} - R^S = \mu_1 - \mu_1\mu_2 - \lambda_2\mu_1 - \mu_2 + \lambda_2^2 + \mu_2^2 - \lambda_1\mu_1 \frac{\lambda_1+\lambda_2+\mu_2}{2\lambda_1+\mu_1} - \lambda_1\lambda_2 + 2\lambda_2\mu_2 - \lambda_1(\lambda_1 + \lambda_2 + \mu_2)\ln\left(\frac{\lambda_2+\mu_2+\lambda_1}{2\lambda_1+\mu_1}\right)$.

Evaluating $R^F_{2M} - R^S$ at $\mu_1 = \frac{1}{\lambda_2}\lambda_1^2 + \frac{\mu_2-\lambda_2}{\lambda_2}\lambda_1$ yields $R^{F-S}(\lambda_1) = \frac{1-2\lambda_2-\mu_2}{\lambda_2}\lambda_1^2 + \frac{\mu_2-\lambda_2+\lambda_2^2-\mu_2^2-\lambda_2\mu_2}{\lambda_2}\lambda_1 + (\lambda_2+\mu_2)^2 - \mu_2 - \lambda_1(\lambda_1 + \lambda_2 + \mu_2)(\ln\lambda_2 - \ln\lambda_1)$, hence $\frac{dR^{F-S}}{d\lambda_1} = 2\frac{1-2\lambda_2-\mu_2}{\lambda_2}\lambda_1 + \frac{\mu_2-\lambda_2+\lambda_2^2-\mu_2^2-\lambda_2\mu_2}{\lambda_2} - \lambda_1\ln\lambda_2 + \lambda_1\ln\lambda_1 - (\lambda_1 + \lambda_2 + \mu_2)(\ln\lambda_2 - \ln\lambda_1 - 1)$, $\frac{d^2R^{F-S}}{d\lambda_1^2} = 2\ln\lambda_1 - 2\ln\lambda_2 + \frac{\lambda_2+\mu_2}{\lambda_1} + 2\frac{1-\mu_2}{\lambda_2} - 1$, $\frac{d^3R^{F-S}}{d\lambda_1^3} = \frac{1}{\lambda_1^2}(2\lambda_1 - \lambda_2 - \mu_2)$.

When $\lambda_2 \le 3\mu_2$, we have that $\overset{\circ}{\lambda}_1 = \frac{1}{2}(\lambda_2+\mu_2)$ and $\varphi''(\overset{\circ}{\lambda}_1) = \frac{1}{\lambda_2}\left(2\lambda_2\ln\frac{\lambda_2+\mu_2}{2\lambda_2} + 2 + \lambda_2 - 2\mu_2\right)$, which is strictly decreasing in μ_2, and at $\mu_2 = \lambda_2$ we find $\varphi''(\overset{\circ}{\lambda}_1) = \frac{1}{\lambda_2}(2-\lambda_2) > 0$

From $\frac{d^3R^{F-S}}{d\lambda_1^3}$ we see that $\lambda_1^m = \max\{\lambda_2 - \mu_2, \frac{1}{2}(\lambda_2+\mu_2)\}$ is the unique minimum point for $\frac{d^2R^{F-S}}{d\lambda_1^2}$ and now we show that $\frac{d^2R^{F-S}(\lambda_1^m)}{d\lambda_1^2} > 0$. In case that $\lambda_1^m = \frac{1}{2}(\lambda_2+\mu_2)$, we find that $\frac{d^2R^{F-S}(\lambda_1^m)}{d\lambda_1^2} = \frac{1}{\lambda_2}\left(2\lambda_2\ln\frac{\lambda_2+\mu_2}{2\lambda_2} + 2 + \lambda_2 - 2\mu_2\right)$, which is strictly decreasing in μ_2. Since $\mu_2 \le \min\{\lambda_2, 1-\lambda_2\}$, we notice that $\frac{d^2R^{F-S}(\lambda_1^m)}{d\lambda_1^2} = \frac{2-\lambda_2}{\lambda_2} > 0$ when $\mu_2 = \lambda_2$, and $\frac{d^2R^{F-S}(\lambda_1^m)}{d\lambda_1^2} = 3 - 2\ln 2\lambda_2 > 0$ when $\mu_2 = 1-\lambda_2$. In case that $\lambda_1^m = \lambda_2 - \mu_2$, which occurs if $\lambda_2 - \mu_2 > \frac{1}{2}(\lambda_2+\mu_2)$, that is if $\frac{1}{3} > \frac{\mu_2}{\lambda_2}$, we obtain $\varphi''(\overset{\circ}{\lambda}_1) = 2\ln\left(1 - \frac{\mu_2}{\lambda_2}\right) + \frac{2}{\lambda_2} + \frac{2\mu_2^2}{\lambda_2(\lambda_2-\mu_2)} > 2\ln\left(1-\frac{\mu_2}{\lambda_2}\right) + \frac{2}{\lambda_2} > 2\ln\frac{2}{3} + 2 > 0$.

Given that R^{F-S} is convex, it follows that R^{F-S} is maximized at $\hat{\lambda}_1 = \max\{0, \lambda_2 - \mu_2\}$ or at $\lambda_1 = \lambda_2$. Since $R^{F-S}(\lambda_2) = 0$, it follows that if $R^{F-S}(\hat{\lambda}_1) \le 0$ then $R^{F-S}(\lambda_1) \le 0$ for each $\lambda_1 \in [\hat{\lambda}_1, \lambda_2]$ and $R^F - R^S \le 0$ for each (λ_1, μ_1), whereas if $R^{F-S}(\hat{\lambda}_1) > 0$ then $R^F - R^S > 0$ for (λ_1, μ_1) close to $(\hat{\lambda}_1, 0)$.

7 Proof of Proposition 5

7.1 Case of $\lambda_2 \le \mu_2$

Let $s_2 = \lambda_2 + \mu_2$, $\alpha = s_2^2 - \mu_2 > 0$, $l_1 = \frac{1}{2}\frac{\alpha^2}{s_2^2}$, $m_1 = \frac{1}{\lambda_2}l_1^2 + \frac{\mu_2-\lambda_2}{\lambda_2}l_1$.

7.1.1 Proof that $R^F - R^S > 0$ for each $(\lambda_1, \mu_1) \in R_{2M}$ such that $\lambda_1 \le l_1$

When $(\lambda_1, \mu_1) \in R_{2M}$,

$$
\begin{aligned}
R^S &= v_L + ((1-\lambda_1)(2 - 2\lambda_2 - \mu_2) - \mu_1(1 - \lambda_2 - \mu_2))\Delta \\
R^F &= v_L + \left(2 - \lambda_1\frac{\lambda_1+\lambda_2+\mu_2}{2\lambda_1+\mu_1}\mu_1 - (2-\lambda_2-\mu_2)(\lambda_1+\lambda_2+\mu_2) - \lambda_1(\lambda_1+\lambda_2+\mu_2)\ln\left(\frac{\lambda_2+\mu_2+\lambda_1}{2\lambda_1+\mu_1}\right)\right)\Delta
\end{aligned}
$$

hence (neglecting Δ)

$$R^F - R^S = \alpha - \lambda_2\lambda_1 + (1-s_2)\,\mu_1 - \lambda_1\frac{\lambda_1+s_2}{2\lambda_1+\mu_1}\mu_1 - \lambda_1(\lambda_1+s_2)\ln\left(\frac{\lambda_1+s_2}{2\lambda_1+\mu_1}\right)$$

We now prove that $R^F > R^S$ if $\lambda_1 \le l_1$ and $\mu_1 = 0$.

- When $\mu_1 = 0$, we have that $R^F > R^S$ boils down to

$$\alpha > \lambda_2\lambda_1 + \lambda_1(\lambda_1+s_2)\ln\left(\frac{1}{2}+\frac{1}{2}\frac{s_2}{\lambda_1}\right) \tag{29}$$

  with $\lambda_1 \le \lambda_2 \le \frac{s_2}{2}$,[52] hence

$$\alpha > \frac{s_2}{2}\lambda_1 + \lambda_1(\lambda_1+s_2)\ln\left(\frac{1}{2}+\frac{1}{2}\frac{s_2}{\lambda_1}\right) \tag{30}$$

  implies (29). Moreover, $\ln(\frac{1}{2}+\frac{1}{2}x) < \sqrt{x-1}$ for each $x > 1$. Therefore, setting $x = \frac{s_2}{\lambda_1}$ we conclude that

$$\alpha > \frac{s_2}{2}\lambda_1 + (\lambda_1+s_2)\sqrt{s_2\lambda_1 - \lambda_1^2} \tag{31}$$

  implies (30). In order for (31) to be satisfied, the right hand side in (31) needs to be smaller than $s_2^2 - \frac{s_2}{2}$ because $\lambda_2 \le \mu_2$ implies $\alpha < s_2^2 - \frac{s_2}{2}$. Since $\left[\frac{s_2}{2}\lambda_1 + (\lambda_1+s_2)\sqrt{s_2\lambda_1-\lambda_1^2}\right]_{\lambda_1=\frac{4}{25}s_2} = \left(\frac{2}{25}+\frac{58}{625}\sqrt{21}\right)s_2^2$ is greater than $s_2^2 - \frac{s_2}{2}$ for each $s_2 \in [\frac{1}{2}, 1]$, and $\frac{s_2}{2}\lambda_1 + (\lambda_1+s_2)\sqrt{s_2\lambda_1-\lambda_1^2}$ is increasing in λ_1, it follows that (31) implies $\lambda_1 < \frac{4}{25}s_2$. Then $\frac{s_2}{2}\lambda_1 + (\lambda_1+s_2)\sqrt{s_2\lambda_1-\lambda_1^2} = \frac{s_2}{2}\sqrt{\lambda_1}\sqrt{\lambda_1} + (\lambda_1+s_2)\sqrt{s_2\lambda_1} < \frac{s_2}{2}\frac{2}{5}\sqrt{s_2}\sqrt{\lambda_1} + (\frac{4}{25}s_2+s_2)\sqrt{s_2\lambda_1} = \frac{1}{5}s_2^{3/2}\sqrt{\lambda_1} + \frac{29}{25}s_2^{3/2}\sqrt{\lambda_1} = \frac{34}{25}s_2^{3/2}\sqrt{\lambda_1} < \frac{34}{25}s_2\sqrt{\lambda_1}$. Hence (31) holds if $\lambda_1 \le l_1$.

- From Lemma 3(ii) we know that $R^F - R^S$ is strictly increasing with respect to μ_1 when $(\lambda_1, \mu_1) \in R_{2M}$. Hence if $(\lambda_1,\mu_1) \in R_{2M}$ with $\lambda_1 \le l_1$ and $\mu_1 > 0$, it follows that $R^F - R^S$ is greater than $R^F - R^S$ when and $\mu_1 = 0$, which we know to be positive.

7.1.2 Proof that $R^F_{1M} - R^S > 0$ for each $(\lambda_1, \mu_1) \in R_{1M}$ such that $\lambda_1 + \mu_1 \le l_1 + m_1$

When $(\lambda_1,\mu_1) \in R_{1M}$, we have that $R^F - R^S$ is Δ times

$$(2\lambda_2+\mu_2-2)\,\rho_1 - (\rho_1^2 + \lambda_2\rho_1 + \mu_2\rho_1)\ln\frac{\lambda_2+\mu_2+\rho_1}{\lambda_1+\mu_1+\rho_1} + (1-\mu_2-2\lambda_2)(\lambda_1+\mu_1) + (1-\lambda_2)\lambda_1 + \alpha \tag{32}$$

We prove that for each $(\lambda_1,\mu_1) \in R_{1M}$ such that $\lambda_1+\mu_1 = l_1+m_1$ we have $R^F - R^S > 0$ and $\frac{\partial(R^F-R^S)}{\partial\mu_1} < 0$.

Precisely, we evaluate $R^F - R^S$ in (32) at $\lambda_1 = 0$, $\mu_1 = l_1 + m_1 = \frac{1}{\lambda_2}l_1^2 + \frac{\mu_2}{\lambda_2}l_1 = \frac{1}{\lambda_2}\left(\frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2}\right)^2 + \frac{\mu_2}{\lambda_2}\frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2}$, with $\rho_1 = \frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2}$. Hence (32) is equal to

$$(2\lambda_2+\mu_2-2)\frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2} + (1-\mu_2-2\lambda_2)\left(\frac{1}{\lambda_2}\left(\frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2}\right)^2 + \frac{\mu_2}{\lambda_2}\frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2}\right)$$
$$+\lambda_2^2 + 2\lambda_2\mu_2 + \mu_2^2 - \mu_2 - \left(\left(\frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2}\right)^2 + \frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)}\right)\left(\ln\frac{2\lambda_2(\lambda_2+\mu_2)^2}{(-\mu_2+\lambda_2^2+\mu_2^2+2\lambda_2\mu_2)^2}\right)$$

[52] Hence $\frac{s_2}{2\lambda_1} \ge 1$, $\frac{1}{2}+\frac{s_2}{2\lambda_1} \ge \frac{3}{2}$, $\ln\left(\frac{1}{2}+\frac{s_2}{2\lambda_1}\right) > \frac{2}{5} > 0$.

which is positive for each (λ_2, μ_2) to the right of the purple curve in the figure. The bold curve is such that $\alpha > 0$ if and only if (λ_2, μ_2) is to the right of the bold curve. Hence $R^F - R^S > 0$ for each (λ_2, μ_2) such that $\alpha > 0$. Therefore $R^F - R^S > 0$ for each $(\lambda_1, \mu_1) \in R_{1M}$ such that $\lambda_1 + \mu_1 = l_1 + m_1$ because we have seen that $R^F - R^S > 0$ at $\lambda_1 = 0$, $\mu_1 = l_1 + m_1$, and as we reduce μ_1 and increase λ_1, keeping $\lambda_1 + \mu_1$ equal to $l_1 + \mu_1$, $R^F - R^S$ increases by (18).

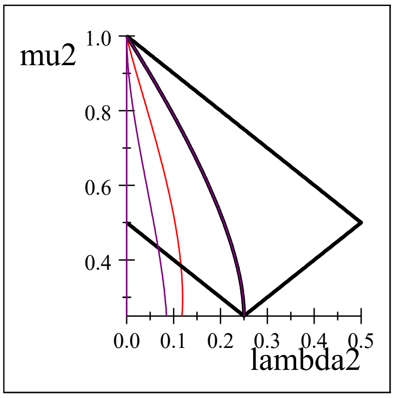
Now we derive $\frac{\partial(R^F - R^S)}{\partial \mu_1}$, which is equal to

$$\frac{(6\lambda_2 + 3\mu_2 - 4)\lambda_2}{4\sqrt{(\lambda_1+\mu_1)\lambda_2 + \frac{1}{4}\mu_2^2}} - \left(\lambda_2 + \frac{\lambda_2^2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2(\lambda_1+\mu_1)}}\right) \ln\left(1 + \frac{2\lambda_2 + 2\mu_2 - 2(\lambda_1+\mu_1)}{2(\lambda_1+\mu_1) - \mu_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + (\lambda_1+\mu_1)\lambda_2}}\right)$$
$$+1 - \frac{3}{2}\lambda_2 - \mu_2$$

Notice that $\frac{\partial(R^F - R^S)}{\partial \mu_1}$ depends on λ_1, μ_1 only through $\lambda_1 + \mu_1$, and $l_1 + m_1 = \frac{1}{\lambda_2} l_1^2 + \frac{\mu_2}{\lambda_2} l_1$, $\sqrt{(l_1+m_1)\lambda_2 + \frac{1}{4}\mu_2^2} = \frac{1}{2}\mu_2 + l_1$, $\rho_1 = l_1$. Then at $(\lambda_1, \mu_1) \in R_{1M}$ such that $\lambda_1 + \mu_1 = l_1 + m_1$ we have

$$\begin{aligned}
\frac{\partial(R^F - R^S)}{\partial \mu_1} &= \frac{(6\lambda_2 + 3\mu_2 - 4)\lambda_2}{4(\frac{1}{2}\mu_2 + l_1)} - \left(\lambda_2 + \frac{\lambda_2^2}{2(\frac{1}{2}\mu_2 + l_1)}\right) \ln\left(\frac{\lambda_2}{l_1}\right) + 1 - \frac{3}{2}\lambda_2 - \mu_2 \\
&= \frac{(6\lambda_2 + 3\mu_2 - 4)\lambda_2}{4(\frac{1}{2}\mu_2 + \frac{((\lambda_2+\mu_2)^2 - \mu_2)^2}{2(\lambda_2+\mu_2)^2})} - \left(\lambda_2 + \frac{\lambda_2^2}{2(\frac{1}{2}\mu_2 + \frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2})}\right) \ln\left(\frac{\lambda_2}{\frac{((\lambda_2+\mu_2)^2-\mu_2)^2}{2(\lambda_2+\mu_2)^2}}\right) + 1 - \frac{3}{2}\lambda_2 - \mu_2
\end{aligned}$$

and $\frac{\partial(R^F - R^S)}{\partial \mu_1} < 0$ at each (λ_1, μ_1) such that $\lambda_1 + \mu_1 = l_1 + m_1$ if and only if (λ_2, μ_2) is to the right of the red curve below, hence $\frac{\partial(R^F - R^S)}{\partial \mu_1} < 0$ at each point such that $\alpha > 0$.



Since $R^F - R^S$ is convex in μ_1, it follows that $\frac{\partial(R^F - R^S)}{\partial \mu_1} < 0$ for each $(\lambda_1, \mu_1) \in R_{1M}$ such that $\lambda_1 + \mu_1 \leq l_1 + m_1$. But $R^F - R^S > 0$ for each $(\lambda_1, \mu_1) \in R_{1M}$ such that $\lambda_1 + \mu_1 = l_1 + m_1$. Therefore $R^F - R^S > 0$ for each $(\lambda_1, \mu_1) \in R_{1M}$ such that $\lambda_1 + \mu_1 \leq l_1 + m_1$.

7.1.3 Conclusion

When $\lambda_2 \leq \mu_2$, the inequality $R^F - R^S > 0$ holds for each (λ_1, μ_1) such that $\lambda_1 \leq l_1$ and $\lambda_1 + \mu_1 \leq l_1 + m_1$.

7.2 Case of $\lambda_2 > \mu_2$

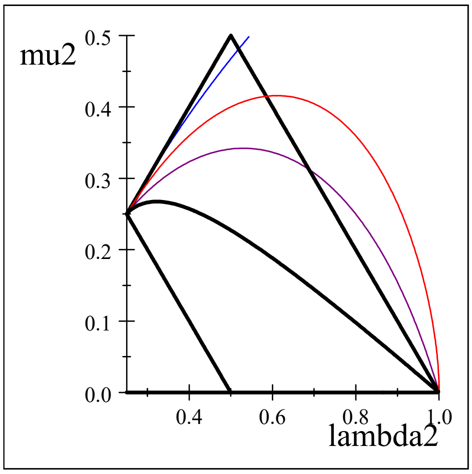
Let $\beta = \mu_2 + 3\lambda_2 - 1 - 2(\lambda_2 - \mu_2)\frac{\lambda_2}{\mu_2} \ln \frac{\lambda_2}{\lambda_2 - \mu_2}$ be equal to (20) on lhs, hence $\beta > 0$, $l_1 = \lambda_2 - \mu_2 + \frac{\mu_2 \beta}{\lambda_2 + 2\lambda_2 \ln \frac{\lambda_2}{\lambda_2 - \mu_2}}$, $m_1 = \frac{1}{\lambda_2} l_1^2 + \frac{\mu_2 - \lambda_2}{\lambda_2} l_1$.

7.2.1 Proof that $R^F - R^S > 0$, for each $(\lambda_1, \mu_1) \in R_{2M}$ such that $\lambda_1 \leq l_1$

Suppose that $\mu_1 = 0$. Then $R^F_{1M} - R^S > 0$ boils down to (13) with $\lambda_2 - \mu_2 < \lambda_1 \leq \lambda_2 \leq \frac{3}{2}s_2 - \frac{1}{2} < s_2$ (the inequality $\lambda_2 \leq \frac{3}{2}s_2 - \frac{1}{2}$ comes from $\lambda_2 + 3\mu_2 - 1 \geq 0$). Notice that $\lambda_2 + (\lambda_1 + s_2)\ln\left(\frac{1}{2} + \frac{s_2}{2\lambda_1}\right)$ is decreasing in λ_1.[53] Hence $\lambda_1 > \lambda_2 - \mu_2$, implies $\lambda_2 + (\lambda_1 + s_2)\ln(\frac{1}{2} + \frac{s_2}{2\lambda_1}) < \lambda_2 + 2\lambda_2 \ln \frac{\lambda_2}{\lambda_2 - \mu_2}$ and $R^F > R^S$ holds if $\lambda_2 - \mu_2 < \lambda_1 \leq \frac{\alpha}{\lambda_2 + 2\lambda_2 \ln \frac{\lambda_2}{\lambda_2 - \mu_2}}$ or equivalently if $\lambda_2 - \mu_2 < \lambda_1 \leq l_1$. Moreover, from Lemma 3(ii) we know that $R^F - R^S$ is strictly increasing with respect to μ_1 when $(\lambda_1, \mu_1) \in R_{2M}$. This implies that $R^F_{2M} > R^S$ holds for each $(\lambda_1, \mu_1) \in R_{2M}$ such that $\lambda_1 \leq l_1$.

7.2.2 Case of $\lambda_2 - \mu_2 - m_1 \geq 0$

The inequality $\lambda_2 - \mu_2 > m_1$ is satisfied if and only if (λ_2, μ_2) lies below the blue curve in the following figure



Proof that $R^F - R^S > 0$ for each (λ_1, μ_1) in the triangle of R_{1MH} with vertices $(\lambda_2 - \mu_2 - m_1, 0)$, $(\lambda_2 - \mu_2 - m_1, m_1)$, $(\lambda_2 - \mu_2, 0)$ when $\lambda_2 - \mu_2 - m_1 \geq 0$ We consider $(\lambda_1, \mu_1) \in R_{1MH}$ and evaluate $R^F - R^S$ at $\lambda_1 = \lambda_2 - \mu_2 - m_1$, $\mu_1 = 0$. Then we obtain

$$
\begin{aligned}
R^F - R^S &= 2 - 2\lambda_2(2 - \lambda_2 - \mu_2) - 2(\lambda_2 - \mu_2)\lambda_2 \ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right) - (2 - 2\lambda_2 - \mu_2 - (1 - \lambda_2 - \mu_2)\lambda_1 - (1 - \lambda_2)\lambda_1) \\
&= \mu_2 - 2\lambda_2(1 - \lambda_2 - \mu_2) - 2(\lambda_2 - \mu_2)\lambda_2 \ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right) + (2 - 2\lambda_2 - \mu_2)\lambda_1 \\
&= \mu_2 - 2\lambda_2(1 - \lambda_2 - \mu_2) - 2(\lambda_2 - \mu_2)\lambda_2 \ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right) + (2 - 2\lambda_2 - \mu_2)\left(\lambda_2 - \mu_2 - \left(\frac{1}{\lambda_2}l_1^2 + \frac{\mu_2 - \lambda_2}{\lambda_2}l_1\right)\right) \\
&= \mu_2 - 2\lambda_2(1 - \lambda_2 - \mu_2) - 2(\lambda_2 - \mu_2)\lambda_2 \ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right) \\
&\quad + (2 - 2\lambda_2 - \mu_2)\left(\lambda_2 - \mu_2 - \frac{1}{\lambda_2}\left(\lambda_2 - \mu_2 + \frac{\mu_2\beta}{\lambda_2(1 + 2\ln\frac{\lambda_2}{\lambda_2 - \mu_2})}\right)^2 - \frac{\mu_2 - \lambda_2}{\lambda_2}\left(\lambda_2 - \mu_2 + \frac{\mu_2\beta}{\lambda_2(1 + 2\ln\frac{\lambda_2}{\lambda_2 - \mu_2})}\right)\right) \\
&= \mu_2\beta\frac{2\mu_2 - 2\lambda_2^2 + 3\lambda_2^3 - 3\mu_2^2 + \mu_2^3 - 6\lambda_2\mu_2 + 4\lambda_2\mu_2^2 + 5\lambda_2^2\mu_2 + 4\left(\ln\frac{\lambda_2}{\lambda_2 - \mu_2}\right)\lambda_2^3\left(\ln\frac{\lambda_2}{\lambda_2 - \mu_2} + 1\right)}{\lambda_2^3\left(2\ln\frac{\lambda_2}{\lambda_2 - \mu_2} + 1\right)^2}
\end{aligned}
$$

[53] Because its derivative is $\ln\left(1 + \frac{s_2}{2\lambda_1} - \frac{1}{2}\right) - \frac{s_2}{\lambda_1} < \frac{s_2}{2\lambda_1} - \frac{1}{2} - \frac{s_2}{\lambda_1} < 0$.

which is positive whenever $\beta > 0$ because numeric analysis shows that $2\mu_2 - 2\lambda_2^2 + 3\lambda_2^3 - 3\mu_2^2 + \mu_2^3 - 6\lambda_2\mu_2 + 4\lambda_2\mu_2^2 + 5\lambda_2^2\mu_2 + 4\left(\ln\frac{\lambda_2}{\lambda_2-\mu_2}\right)\lambda_2^3\left(\ln\frac{\lambda_2}{\lambda_2-\mu_2}+1\right) > 0$ for each (λ_2,μ_2) such that $\beta > 0$.

If λ_1 is raised above $\lambda_2 - \mu_2 - m_1$ (still with $\mu_1 = 0$) till $\lambda_2 - \mu_2$, then $R^F - R^S$ increases, hence $R^F - R^S > 0$. Moreover, if μ_1 is raised above from 0 till the diagonal border of R_{1MH} is reached, then $R^F - R^S$ increases, hence remains positive and $R^F - R^S > 0$ for each point in the triangle R_{1MH} with vertices $(\lambda_2 - \mu_2 - m_1, 0)$, $(\lambda_2 - \mu_2 - m_1, m_1)$, $(\lambda_2 - \mu_2, 0)$.

Proof that $R^F - R^S > 0$ for each $(\lambda_1, \mu_1) \in R_{1M}$ such that $\lambda_2 - \mu_2 - m_1 \leq \lambda_1$, $\lambda_1 + \mu_1 \leq l_1 + m_1$ when $\lambda_2 - \mu_2 - m_1 \geq 0$ We evaluate $\frac{\partial(R^F-R^S)}{\partial\mu_1}$ at $(\lambda_1,\mu_1) = (l_1, m_1)$, using $l_1 + m_1 = \frac{1}{\lambda_2}l_1^2 + \frac{\mu_2}{\lambda_2}l_1$ and $\sqrt{(l_1+m_1)\lambda_2 + \frac{1}{4}\mu_2^2} = \frac{1}{2}\mu_2 + l_1 = \lambda_2 - \frac{1}{2}\mu_2 + \frac{\mu_2\left(\mu_2+3\lambda_2-1-2(\lambda_2-\mu_2)\frac{\lambda_2}{\mu_2}\ln\frac{\lambda_2}{\lambda_2-\mu_2}\right)}{\lambda_2(1+2\ln\frac{\lambda_2}{\lambda_2-\mu_2})}$, $\rho_1 = l_1$. Then

$$
\begin{aligned}
\frac{\partial(R^F-R^S)}{\partial\mu_1} &= \frac{(6\lambda_2+3\mu_2-4)\lambda_2}{4(\lambda_2-\frac{1}{2}\mu_2+\frac{\mu_2\beta}{\lambda_2(1+2\ln\frac{\lambda_2}{\lambda_2-\mu_2})})} + 1 - \frac{3}{2}\lambda_2 - \mu_2 \\
&\quad - \left(\lambda_2 + \frac{\lambda_2^2}{2(\lambda_2-\frac{1}{2}\mu_2+\frac{\mu_2\beta}{\lambda_2(1+2\ln\frac{\lambda_2}{\lambda_2-\mu_2})})}\right)\ln\left(\lambda_2^2\frac{2\ln\frac{\lambda_2}{\lambda_2-\mu_2}+1}{\lambda_2^2+\mu_2^2+2\lambda_2\mu_2-\mu_2}\right)
\end{aligned}
$$

and $\frac{\partial(R^F-R^S)}{\partial\mu_1} < 0$ for each λ_2, μ_2 such that $\beta > 0$.

We evaluate $R^F - R^S$ at $\lambda_1 = \lambda_2 - \mu_2 - m_1$, $\mu_1 = \mu_2 - \lambda_2 + l_1 + 2m_1$, so that $\lambda_1 + \mu_1 = l_1 + m_1$, $\sqrt{\frac{1}{4}\mu_2 + \lambda_2(\lambda_1+\mu_1)} = \frac{1}{2}\mu_2 + l_1$, $\rho_1 = l_1$. Hence $R^F - R^S$ is equal to the following, in which $D = \lambda_2 + 2\lambda_2\ln\frac{\lambda_2}{\lambda_2-\mu_2}$:

$$
\begin{aligned}
& (2\lambda_2+\mu_2-2)\,l_1 - (l_1^2+\lambda_2 l_1+\mu_2 l_1)\ln\frac{\lambda_2}{l_1} + (1-\mu_2-2\lambda_2)\,(\frac{1}{\lambda_2}l_1^2+\frac{\mu_2}{\lambda_2}l_1) + (1-\lambda_2)\,l_1 + (\lambda_2+\mu_2)^2 - \mu_2 \\
= \;& -(1-\lambda_2-\mu_2)\,l_1 - (l_1^2+(\lambda_2+\mu_2)l_1)\ln\frac{\lambda_2}{l_1} + (1-\mu_2-2\lambda_2)\,(\frac{1}{\lambda_2}l_1^2+\frac{\mu_2}{\lambda_2}l_1) + (\lambda_2+\mu_2)^2 - \mu_2 \\
= \;& l_1\left(-(1-\lambda_2-\mu_2) - (l_1+\lambda_2+\mu_2)\ln\frac{\lambda_2}{l_1} + \frac{1-\mu_2-2\lambda_2}{\lambda_2}(l_1+\mu_2)\right) + \frac{(\lambda_2+\mu_2)^2-\mu_2}{a} \\
= \;& l_1\left(\frac{1-\mu_2-2\lambda_2}{\lambda_2}\frac{\mu_2\beta}{D} - \left(\frac{\mu_2\beta}{D}+2\lambda_2\right)\ln\frac{\lambda_2 D}{\alpha} + 2\lambda_2\ln\frac{\lambda_2}{\lambda_2-\mu_2}\right) \\
= \;& l_1\left(\frac{\mu_2\beta}{D}\left(\frac{1-\mu_2-2\lambda_2}{\lambda_2} - \ln\left(1+\frac{\mu_2(D-\beta)}{\mu_2\beta+(\lambda_2-\mu_2)D}\right)\right) + 2\lambda_2\ln\left(1+\frac{\mu_2\beta}{(\lambda_2-\mu_2)D}\right)\right)
\end{aligned}
$$

It is immediate that this expression is zero when $\beta = 0$, and numeric analysis shows it is positive if and only if $\beta > 0$.

Conclusion for the case of $\lambda_2 > \mu_2$ and $\lambda_2 - \mu_2 - m_1 \geq 0$ $R^F > R^S$ in the trapezoid with vertices $(\lambda_2-\mu_2-m_1, 0)$, $(\lambda_2-\mu_2-m_1, \mu_2-\lambda_2+l_1+2m_1)$, (l_1, m_1), $(l_1, 0)$, because starting from borders of R_{2M} or R_{1MH} and increasing μ_1 decreases $R^F - R^S$, but $R^F - R^S > 0$ at the diagonal edge of the trapezoid.

7.2.3 Case of $\lambda_2 - \mu_2 - m_1 < 0$

Now suppose that (λ_2, μ_2) lies above the blue curve in the above figure, so that $\lambda_2 - \mu_2 - m_1 < 0$.

Proof that $R^F - R^S > 0$ for each $(\lambda_1, \mu_1) \in R_{1MH}$ when $\lambda_2 - \mu_2 - m_1 < 0$ We consider (λ_1, μ_1) and evaluate

$$R^F - R^S = \left(2 - 2\lambda_2(2 - \lambda_2 - \mu_2) - 2(\lambda_2 - \mu_2)\lambda_2 \ln\left(\frac{\lambda_2}{\lambda_2 - \mu_2}\right)\right) - (2 - 2\lambda_2 - \mu_2 - (2 - 2\lambda_2 - \mu_2)\lambda_1 - (1 - \lambda_2 - \mu_2)\mu_1)$$

at $\lambda_1 = 0$, $\mu_1 = 0$. Then $R^F - R^S = \mu_2 - 2\lambda_2 + 2\lambda_2^2 + 2\lambda_2\mu_2 - 2\lambda_2(\lambda_2 - \mu_2)\ln\frac{\lambda_2}{\lambda_2 - \mu_2}$, which is positive above the red curve. In particular, $R^F - R^S > 0$ for λ_2, μ_2 such that $\lambda_2 - \mu_2 < m_1$. Then $R^F - R^S > 0$ in the whole triangle R_{1MH}.

Proof that $R^F - R^S > 0$ for each $(\lambda_1, \mu_1) \in R_{1M}$ such that $\lambda_1 + \mu_1 \le l_1 + m_1$ when $\lambda_2 - \mu_2 - m_1 < 0$ We consider $(\lambda_1, \mu_1) \in R_{1M}$ and evaluate $R^F - R^S$ from (32) at $(\lambda_1, \mu_1) = (0, l_1 + m_1)$, hence $l_1 + m_1 = \frac{1}{\lambda_2} l_1^2 + \frac{\mu_2}{\lambda_2} l_1$ and $\rho_1 = l_1$. Thus $R^F - R^S$ is

$$\left(\lambda_2 - \mu_2 + \frac{\mu_2\beta}{\lambda_2(1 + 2\ln\frac{\lambda_2}{\lambda_2 - \mu_2})}\right)\left(2\lambda_2 + \mu_2 - 2 - \left(2\lambda_2 + \frac{\mu_2\beta}{\lambda_2(1 + 2\ln\frac{\lambda_2}{\lambda_2 - \mu_2})}\right)\ln\lambda_2^2 \frac{1 + 2\ln\frac{\lambda_2}{\lambda_2 - \mu_2}}{(\lambda_2 + \mu_2)^2 - \mu_2}\right)$$
$$+ (1 - \mu_2 - 2\lambda_2)\left(\frac{1}{\lambda_2}\left(\lambda_2 - \mu_2 + \frac{\mu_2\beta}{\lambda_2(1 + 2\ln\frac{\lambda_2}{\lambda_2 - \mu_2})}\right)^2 + \frac{\mu_2}{\lambda_2}\left(\lambda_2 - \mu_2 + \frac{\mu_2\beta}{\lambda_2(1 + 2\ln\frac{\lambda_2}{\lambda_2 - \mu_2})}\right)\right) + (\lambda_2 + \mu_2)^2 - \mu_2$$

and it is positive if and only if (λ_2, μ_2) lies above the purple curve in the figure above. Hence it is positive if (λ_2, μ_2) is above the blue curve

Conclusion for the case of $\lambda_2 > \mu_2$ and $\lambda_2 - \mu_2 - m_1 < 0$ $R^F > R^S$ at $(\lambda_1, \mu_1) = (0, l_1 + m_1)$ (just found) and in each point on the segment which connects $(0, l_1 + m_1)$ to $(\lambda_1, \mu_1) = (l_1, m_1)$ because $\lambda_1 + \mu_1$ is fixed, but $R^F - R^S$ is increasing in λ_1 and less increasing in μ_1. Hence $R^F > R^S$ holds in the trapezoid with vertices $(0,0), (0, l_1 + m_1), (l_1, m_1), (l_1, 0)$.

The trapzoid with the following vertices, with $\gamma = \max\{\lambda_2 - \mu_2 - m_1, 0\}$,

$$(\gamma, 0), (\gamma, l_1 + m_1 - \gamma), (l_1, m_1), (l_1, 0)$$

captures the case of $\lambda_2 - \mu_2 - m_1 \ge 0$ and of $\lambda_2 - \mu_2 - m_1 < 0$ in a single writing.

7.3 Proof of Proposition 7

Step 0: When $(\lambda_1, \mu_1) \in R_{1M}$, R^F is strictly convex in μ_1 From (24) we see that R^F is equal to v_L plus Δ times $2 - (\rho_1 + \lambda_2 + \mu_2) - \rho_1\lambda_2 \frac{1 - \frac{\rho_1}{\lambda_1 + \mu_1}}{\frac{\rho_1}{\lambda_1 + \mu_1}} - (\rho_1^2 + \lambda_2\rho_1 + \mu_2\rho_1)\ln\frac{\left(1 + \frac{\rho_1}{\lambda_2 + \mu_2}\right)\frac{\rho_1}{\lambda_1 + \mu_1}}{\frac{\rho_1}{\lambda_2 + \mu_2}\left(1 + \frac{\rho_1}{\lambda_1 + \mu_1}\right)} - \frac{\rho_1 + \lambda_2 + \mu_2}{1 + \frac{\rho_1}{\lambda_2 + \mu_2}}\left(1 - (\rho_1 + \lambda_2 + \mu_2) + \frac{\rho_1}{\lambda_2 + \mu_2}\right)$ and subtracting R^S we obtain (neglecting the factor Δ) $(2\lambda_2 + \mu_2 - 2)\rho_1 - (\rho_1^2 + \lambda_2\rho_1 + \mu_2\rho_1)(\ln(\lambda_2 + \mu_2 + \rho_1) - \ln(\lambda_1 + \mu_1 +$ $2 + \lambda_2^2 - 2\mu_2 - 2\lambda_2 + \mu_2^2 - \lambda_1\lambda_2 - \lambda_2\mu_1 + 2\lambda_2\mu_2 - (2 - 2\lambda_2 - \mu_2)(1 - \lambda_1) + (1 - \lambda_2 - \mu_2)\mu_1$. Hence the derivative of R^F with respect to μ_1 (omitting the factor Δ) is

$$\frac{\partial R^F}{\partial \mu_1} = (2\lambda_2 + \mu_2 - 2)\rho_1' - (2\rho_1\rho_1' + \lambda_2\rho_1' + \mu_2\rho_1')(\ln(\lambda_2 + \mu_2 + \rho_1) - \ln(\lambda_1 + \mu_1 + \rho_1))$$
$$-(\rho_1^2 + \lambda_2\rho_1 + \mu_2\rho_1)\left(\frac{\rho_1'}{\lambda_2 + \mu_2 + \rho_1} - \frac{1 + \rho_1'}{\lambda_1 + \mu_1 + \rho_1}\right) - \lambda_2$$

and since $\rho_1 = \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2$, with $s = \lambda_1 + \mu_1$, and $\rho_1' = \frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2+\lambda_2 s}}$, it follows that

$$\begin{aligned}
\frac{\partial R^F}{\partial \mu_1} &= (2\lambda_2 + \mu_2 - 2)\frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}} - \left(\lambda_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + s\lambda_2}\right)\frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}} \ln \frac{\lambda_2 + \mu_2 + \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2}{s + \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2} \\
&\quad - \left(\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2\right)\left(\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} + \lambda_2 + \frac{1}{2}\mu_2\right)\left(\frac{\frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2+\lambda_2 s}}}{\lambda_2 + \mu_2 + \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2} - \frac{1 + \frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2+\lambda_2 s}}}{s + \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2}\right) \\
&= (2\lambda_2 + \mu_2 - 2)\frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}} - \left(\lambda_2 + \frac{\lambda_2^2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}}\right)\ln\left(1 + \frac{2\lambda_2 + 2\mu_2 - 2s}{2s - \mu_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + s\lambda_2}}\right) - \frac{3}{2}\lambda_2 \\
&\quad + \frac{\lambda_2}{2}\frac{\mu_2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}} + \left(\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2\right)\left(\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} + \lambda_2 + \frac{1}{2}\mu_2\right)\frac{1 + \frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2+\lambda_2 s}}}{s + \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2} \\
&= \frac{1}{4}\lambda_2\frac{4\lambda_2 + 3\mu_2 - 4}{\sqrt{s\lambda_2 + \frac{1}{4}\mu_2^2}} - \left(\lambda_2 + \frac{\lambda_2^2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}}\right)\ln\left(1 + \frac{2\lambda_2 + 2\mu_2 - 2s}{2s - \mu_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + s\lambda_2}}\right) \\
&\quad - \frac{3}{2}\lambda_2 + \lambda_2\left(s + \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2\right)\frac{1 + \frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2+\lambda_2 s}}}{s + \sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s} - \frac{1}{2}\mu_2} \\
&= \frac{(6\lambda_2 + 3\mu_2 - 4)\lambda_2}{4\sqrt{s\lambda_2 + \frac{1}{4}\mu_2^2}} - \frac{1}{2}\lambda_2 - \left(\lambda_2 + \frac{\lambda_2^2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}}\right)\ln\left(1 + \frac{2\lambda_2 + 2\mu_2 - 2s}{2s - \mu_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + s\lambda_2}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 R^F}{\partial \mu_1^2} &> -\frac{(6\lambda_2 + 3\mu_2 - 4)\lambda_2^2}{8\left(s\lambda_2 + \frac{1}{4}\mu_2^2\right)^{3/2}} + 2\left(\lambda_2 + \frac{\lambda_2^2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}}\right)\frac{1}{1 + \frac{2\lambda_2 + 2\mu_2 - 2s}{2s - \mu_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + s\lambda_2}}}\frac{1}{2s - \mu_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + s\lambda_2}} \\
&= \lambda_2\left(-\frac{(6\lambda_2 + 3\mu_2 - 4)\lambda_2}{8\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}^3} + 2\left(1 + \frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}}\right)\frac{1}{2\lambda_2 + \mu_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + s\lambda_2}}\right) \\
&= \frac{\lambda_2}{\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}}\left(-\frac{(6\lambda_2 + 3\mu_2 - 4)\lambda_2}{2\mu_2^2 + 8s\lambda_2} + \frac{\lambda_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}}{2\lambda_2 + \mu_2 + 2\sqrt{\frac{1}{4}\mu_2^2 + s\lambda_2}}\right) \\
&= \lambda_2\frac{2\sqrt{s\lambda_2 + \frac{1}{4}\mu_2^2}\left(8\lambda_2 s + 4\lambda_2 - 3\lambda_2\mu_2 - 6\lambda_2^2 + 2\mu_2^2\right) + \lambda_2\left(8\lambda_2 s + 8\lambda_2 - 12\lambda_2\mu_2 - 12\lambda_2^2 - \mu_2^2 + 4\mu_2\right)}{\sqrt{\frac{1}{4}\mu_2^2 + \lambda_2 s}\,2\left(2\lambda_2 + \mu_2 + 2\sqrt{s\lambda_2 + \frac{1}{4}\mu_2^2}\right)(4s\lambda_2 + \mu_2^2)}
\end{aligned}$$

To conclude, we prove that $2\sqrt{s\lambda_2 + \frac{1}{4}\mu_2^2}\left(8\lambda_2 s + 4\lambda_2 - 3\lambda_2\mu_2 - 6\lambda_2^2 + 2\mu_2^2\right) + \lambda_2\left(8\lambda_2 s + 8\lambda_2 - 12\lambda_2\mu_2 - 12\lambda_2^2 - \mu_2^2 + 4\mu\right.$ 0. First we show that $8\lambda_2 s + 4\lambda_2 - 3\lambda_2\mu_2 - 6\lambda_2^2 + 2\mu_2^2 > 0$. We know that $8\lambda_2 s + 4\lambda_2 - 3\lambda_2\mu_2 - 6\lambda_2^2 + 2\mu_2^2 > 8\lambda_2(\lambda_2 - \mu_2) + 4\lambda_2 - 3\lambda_2\mu_2 - 6\lambda_2^2 + 2\mu_2^2 = 4\lambda_2 - 11\lambda_2\mu_2 + 2\lambda_2^2 + 2\mu_2^2$, which is decreasing in μ_2 (since $-11\lambda_2 + 4\mu_2 < 0$), hence $4\lambda_2 - 11\lambda_2\mu_2 + 2\lambda_2^2 + 2\mu_2^2 > 4\lambda_2 - 11\lambda_2^2 + 2\lambda_2^2 + 2\lambda_2^2 = \lambda_2(4 - 7\lambda_2) > 0$ (for $\lambda_2 < \frac{1}{2}$) and $4\lambda_2 - 11\lambda_2\mu_2 + 2\lambda_2^2 + 2\mu_2^2 > 4\lambda_2 - 11\lambda_2(1 - \lambda_2) + 2\lambda_2^2 + 2(1 - \lambda_2)^2 = (3\lambda_2 - 1)(5\lambda_2 - 2) > 0$ for $\lambda_2 > \frac{1}{2}$.

Finally, since $8\lambda_2 s+4\lambda_2-3\lambda_2\mu_2-6\lambda_2^2+2\mu_2^2>0$ it follows that $2\sqrt{s\lambda_2+\frac{1}{4}\mu_2^2}\left(8\lambda_2 s+4\lambda_2-3\lambda_2\mu_2-6\lambda_2^2+2\mu_2^2\right)$ $+\lambda_2\left(8\lambda_2 s+8\lambda_2-12\lambda_2\mu_2-12\lambda_2^2-\mu_2^2+4\mu_2\right)$ is increasing in s. We evaluate this expression at $s=\lambda_2-\mu_2$, and we find $2\left(\lambda_2-\frac{1}{2}\mu_2\right)\left(8\lambda_2(\lambda_2-\mu_2)+4\lambda_2-3\lambda_2\mu_2-6\lambda_2^2+2\mu_2^2\right)+\lambda_2\left(8\lambda_2(\lambda_2-\mu_2)+8\lambda_2-12\lambda_2\mu_2-12\lambda_2^2-\mu_2^2+4\mu_2\right)$ $-44\lambda_2^2\mu_2+16\lambda_2^2+14\lambda_2\mu_2^2-2\mu_2^3$. This is decreasing in μ_2 because $-44\lambda_2^2+28\lambda_2\mu_2-6\mu_2^2<-44\lambda_2^2+28\lambda_2^2-6\mu_2^2=-16\lambda_2^2-6\mu_2^2<0$. Hence $-44\lambda_2^2\lambda_2+16\lambda_2^2+14\lambda_2\lambda_2^2-2\lambda_2^3=16\lambda_2^2(1-2\lambda_2)>0$ for $\lambda_2<\frac{1}{2}$, $-44\lambda_2^2(1-\lambda_2)+16\lambda_2^2+14\lambda_2(1-\lambda_2)^2-2(1-\lambda_2)^3=2(5\lambda_2-1)(2\lambda_2-1)(3\lambda_2-1)>0$ for $\lambda_2>\frac{1}{2}$.

Step 1: The case of $\lambda_2\le\mu_2$, that is $\lambda_2\le\frac{1}{2}$ Suppose that $\lambda_2\le\mu_2$. We consider $(\lambda_1,\mu_1)=(\alpha\lambda_2,\alpha\mu_2)$ and prove that there exists $\alpha^*\in(0,1)$ such that $R^F>R^S$ if and only if $\alpha\in[0,\alpha^*)$.

Let $\gamma(\alpha)=R^F-R^S$. We know from Proposition 4(i) that $\gamma(0)>0$. We prove below that $\gamma(\alpha)<0$ for α close to 1, hence the set $\{\alpha\in(0,1):\gamma(\alpha)=0\}$ is non-empty. We show moreover that γ is strictly convex, which implies that the set $\{\alpha\in(0,1):\gamma(\alpha)=0\}$ consists of a single element: if we let $\alpha^*=\min\{\alpha\in(0,1):\gamma(\alpha)=0\}$, it follows that $\gamma(\alpha)>0$ for each $\alpha\in(0,\alpha^*)$ by definition of α^*, and $\gamma(\alpha)<0$ for each $\alpha\in(\alpha^*,1)$ since γ strictly convex implies $\gamma(\alpha)<\frac{1-\alpha}{1-\alpha^*}\gamma(\alpha^*)+\frac{\alpha-\alpha^*}{1-\alpha^*}\gamma(1)$ and the right hand side is zero since $\gamma(\alpha^*)=0$ by definition of α^*, $\gamma(1)=0$ because $\alpha=1$ implies $(\lambda_1,\mu_1)=(\lambda_2,\mu_2)$.

In order to see that $\gamma(\alpha)<0$ for α close to 1, we use $\gamma(1)=0$ and show below that $\gamma'(1)>0$; thus $\gamma(\alpha)<0$ for α close to 1 follows. Recall from Lemma 3(i) that when $(\lambda_1,\mu_1)\in R_{1M}$, R^F depends on λ_1,μ_1 only through $\lambda_1+\mu_1$. Hence $\frac{\partial R^F}{\partial\lambda_1}=\frac{\partial R^F}{\partial\mu_1}$, and given $(\lambda_1,\mu_1)=(\alpha\lambda_2,\alpha\mu_2)$ we find that $\frac{dR^F}{d\alpha}=\lambda_2\frac{\partial R^F}{\partial\lambda_1}+\mu_2\frac{\partial R^F}{\partial\mu_1}=\frac{\partial R^F}{\partial\mu_1}$ since $\lambda_2+\mu_2=1$. We prove below that (omitting the irrelevant factor $\Delta>0$)

$$\begin{aligned}\frac{\partial R^F}{\partial\mu_1} &= (2\lambda_2+\mu_2-2)\rho_1'-(2\rho_1\rho_1'+\lambda_2\rho_1'+\mu_2\rho_1')(\ln(\lambda_2+\mu_2+\rho_1)-\ln(\lambda_1+\mu_1+\rho_1))\\ &\quad -(\rho_1^2+\lambda_2\rho_1+\mu_2\rho_1)\left(\frac{\rho_1'}{\lambda_2+\mu_2+\rho_1}-\frac{1+\rho_1'}{\lambda_1+\mu_1+\rho_1}\right)-\lambda_2\end{aligned}$$

with $\rho_1=\sqrt{\frac{1}{4}\mu_2^2+\lambda_2(\lambda_1+\mu_1)}-\frac{1}{2}\mu_2$ and $\rho_1'=\frac{\lambda_2}{2\sqrt{\frac{1}{4}\mu_2^2+\lambda_2(\lambda_1+\mu_1)}}$. At $\alpha=1$ we have $\lambda_1+\mu_1=1$ and $\rho_1=\lambda_2$, $\rho_1'=\frac{\lambda_2}{1+\lambda_2}$, hence $\frac{\partial R^F}{\partial\mu_1}=-\lambda_2\frac{1-\lambda_2}{1+\lambda_2}$. From (17) it follows that $\frac{dR^S}{d\alpha}=-\lambda_2(1-\lambda_2)$. Finally, $\gamma'(1)=\frac{dR^F}{d\alpha}-\frac{dR^S}{d\alpha}=\frac{\partial R^F}{\partial\mu_1}-\frac{dR^S}{d\alpha}=\lambda_2^2\frac{1-\lambda_2}{1+\lambda_2}>0$.

Step 2: The case of $\lambda_2>\mu_2$, that is $\lambda_2>\frac{1}{2}$ Suppose that $\lambda_2>\mu_2$. We consider $(\lambda_1,\mu_1)=(\alpha\lambda_2,\alpha\mu_2)$ and prove that there exists an interval I, including $\alpha=2\lambda_2-1$, such that $R^F>R^S$ if and only if $\alpha\in I$.

Let $\gamma(\alpha)=R^F-R^S$. First notice that $\alpha=2\lambda_2-1$ is such that $\lambda_1+\mu_1=\lambda_2-\mu_2$ (that is, $\lambda_1+\mu_1=2\lambda_2-1$). This means that when $\alpha=2\lambda_2-1$, (λ_1,μ_1) lies on the boundary between R_{1MH} and R_{1M}: see Figure 1b. Thus (17) and (25) reveal that $\gamma(2\lambda_2-1)$ is equal to Δ times $(1-\lambda_2)\left(1-\lambda_2+2\lambda_2^2\right)-2\lambda_2(2\lambda_2-1)\ln\left(1+\frac{1-\lambda_2}{2\lambda_2-1}\right)$. We prove below that $\gamma(2\lambda_2-1)>0$ for each $\lambda_2\in(\frac{1}{2},1)$.

- For $\lambda_2\in(\frac{1}{2},\frac{16}{25}]$ we use the inequality $\ln(1+z)<\sqrt{z}$.[54] Thus

$$\begin{aligned}\gamma(2\lambda_2-1) &> (1-\lambda_2)\left(1-\lambda_2+2\lambda_2^2\right)-2\lambda_2(2\lambda_2-1)\sqrt{\frac{1-\lambda_2}{2\lambda_2-1}}\\ &= 2\lambda_2\sqrt{1-\lambda_2}\left(\sqrt{1-\lambda_2}\left(\lambda_2+\frac{1}{2\lambda_2}-\frac{1}{2}\right)-\sqrt{2\lambda_2-1}\right)\end{aligned}$$

[54] This inequality holds for each $z>0$, as it holds with equality when $z=0$ and the derivative of $\sqrt{z}-\ln(1+z)$ is $\frac{(1-\sqrt{z})^2}{2\sqrt{z}(1+z)}\ge0$. Finally, notice that $\frac{1-\lambda_2}{2\lambda_2-1}$ in $\ln\left(1+\frac{1-\lambda_2}{2\lambda_2-1}\right)$ is positive since $\lambda_2>\frac{1}{2}$.

and $\sqrt{1-\lambda_2}\left(\lambda_2+\frac{1}{2\lambda_2}-\frac{1}{2}\right)-\sqrt{2\lambda_2-1}$ is decreasing in the interval $(\frac{1}{2},\frac{16}{25}]$, is positive at $\lambda_2=\frac{16}{25}$. Therefore it is positive and $\gamma(2\lambda_2-1)>0$ for each $\lambda_2\in(\frac{1}{2},\frac{16}{25}]$.

- For $\lambda_2\in(\frac{16}{25},1)$ we use the inequality $\ln(1+z)<z-\frac{1}{2}z^2+\frac{1}{3}z^3$, which holds for each $z>0$ by Taylor's theorem. Hence $\gamma(2\lambda_2-1)>(1-\lambda_2)\left(1-\lambda_2+2\lambda_2^2\right)-2\lambda_2(2\lambda_2-1)\left(\frac{1-\lambda_2}{2\lambda_2-1}-\frac{1}{2}\left(\frac{1-\lambda_2}{2\lambda_2-1}\right)^2+\frac{1}{3}\left(\frac{1-\lambda_2}{2\lambda_2-1}\right)^3\right)=\frac{(1-\lambda_2)^3}{3(2\lambda_2-1)^2}\left(24\lambda_2^2-20\lambda_2+3\right)$. The latter expression is positive for each $\lambda_2\in(\frac{16}{25},1)$, therefore $\gamma(2\lambda_2-1)>0$ for each $\lambda_2\in(\frac{16}{25},1)$.

After establishing that $\gamma(2\lambda_2-1)>0$, we notice that the set of $\alpha\in(0,2\lambda_2-1)$ such that $\gamma(\alpha)>0$ is an interval (with upper bound $2\lambda_2-1$) since γ is linear in α when $\alpha\in(0,2\lambda_2-1)$ as $(\lambda_1,\mu_1)=(\alpha\lambda_2,\alpha\mu_2)\in R_{1MH}$ and R^F is constant with respect to α, R^S is linear in λ_1,μ_1. The set of $\alpha\in(2\lambda_2-1,1)$ such that $\gamma(\alpha)>0$ is an interval (with lower bound $2\lambda_2-1$) as R^F-R^S is convex in α, by the same argument given in the proof for Step 1. In particular, when $\alpha\in(0,2\lambda_2-1)$ we have that $\gamma(\alpha)$ turns out to be equal to, from (17) and (25), $2-2\lambda_2-2\lambda_2(2\lambda_2-1)\ln\frac{\lambda_2}{2\lambda_2-1}-(1-\lambda_2)(1-\alpha\lambda_2)$, and this is positive if and only if $\alpha>\frac{\lambda_2-1+2\lambda_2(2\lambda_2-1)\ln\left(1+\frac{1-\lambda_2}{2\lambda_2-1}\right)}{\lambda_2(1-\lambda_2)}$. Finally, we notice that the latter right hand side tends to 1 as $\lambda_2\to1$. ■