UNIVERSITÀ DEGLI STUDI FIRENZE
DISEI
DIPARTIMENTO DI SCIENZE PER L'ECONOMIA E L'IMPRESA

Working Papers
Quantitative Methods for Social Sciences

Manipulation of positional social choice correspondences under incomplete information

Raffaele Berzi, Daniela Bubboloni, Michele Gori

Working Paper N. 02/2025

DISEI, Università degli Studi di Firenze
Via delle Pandette 9, 50127 Firenze (Italia) www.disei.unifi.it

Manipulation of positional social choice correspondences under incomplete information

Raffaele Berzi
Dipartimento di Scienze per l'Economia e l'Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127, Firenze, Italy
e-mail: raffaele.berzi@unifi.it
https://orcid.org/0009-0008-1489-6975

Daniela Bubboloni
Dipartimento di Matematica e Informatica U.Dini
Università degli Studi di Firenze
viale Morgagni 67/a, 50134 Firenze, Italy
e-mail: daniela.bubboloni@unifi.it
https://orcid.org/0000-0002-1639-9525

Michele Gori
Dipartimento di Scienze per l'Economia e l'Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127, Firenze, Italy
e-mail: michele.gori@unifi.it
https://orcid.org/0000-0003-3274-041X

March 29, 2025

Abstract

We study the manipulability of social choice correspondences in situations where individuals have incomplete information about others' preferences. We propose a general concept of manipulability that depends on the extension rule used to derive preferences over sets of alternatives from preferences over alternatives, as well as on individuals' level of information. We then focus on the manipulability of social choice correspondences when the Kelly extension rule is used, and individuals are assumed to have the capability to anticipate the outcome of the collective decision. Under these assumptions, we introduce some monotonicity properties of social choice correspondences whose combined satisfaction is sufficient for manipulability, prove a result of manipulability for unanimous positional social choice correspondences, and present a detailed analysis of the manipulability properties for the Borda, the Plurality and the Negative Plurality social choice correspondences.

Keywords: Social choice correspondence; Manipulability; Strategy-proofness; Extension rule; Incomplete information.

JEL classification: D71, D72.

MSC classification: 91B12, 91B14.

1 Introduction

Consider a group of individuals who must select one or more alternatives among the ones in a given set. Assume that such a selection must be based solely on individuals' preferences, expressed through rankings of alternatives. Any procedure that associates a non-empty set of alternatives with each preference profile, that is, a complete list of individual preferences, is called a social choice correspondence (SCC). A SCC is manipulable if there are situations where an individual has an incentive to misrepresent her preferences because doing so makes the SCC produce an outcome she prefers more; a SCC is strategy-proof if it is not manipulable. If a SCC is resolute, which means that it always selects a singleton, there is no ambiguity in understanding whether, for a certain individual, an outcome is better than another. Indeed, that fact can be naturally deduced by her preferences. The well-known Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) shows that any resolute SCC must be manipulable, provided that the alternatives are at least three, each alternative can be potentially an outcome of the SCC, and the SCC is not dictatorial.

When a SCC is not resolute, the definition of strategy-proofness depends on how the outcomes of the SCCs, which are, in principle, sets of alternatives of any size, are compared by individuals. The way an individual compares sets of alternatives clearly depends on her preferences over alternatives, but there are several reasonable possibilities to specify that dependence. In other words, it is possible to figure out a variety of reasonable extension rules, namely mechanisms that associate with any preference relation on the set of alternatives a preference relation on the set of the nonempty sets of alternatives.[1] The use of different extension rules has led to different definitions of strategy-proofness and several impossibility results have been proved (Pattanaik 1975; Gärdenfors, 1976; Kelly, 1977; Barberà, 1977a, 1977b; Duggan and Schwartz, 2000; Barberà and Dutta, 2001; Taylor, 2002; Ching and Zhou, 2002; Sato, 2008; for a survey, see also Taylor, 2005). As observed by Barberà (2011), most contributions establish results for SCCs that are analogous to the classic impossibility theorem for resolute SCCs. That suggests that, even when the assumption of resoluteness is removed, there is still no significant room for strategy-proofness.

In this paper, with the aim of finding positive results, we analyze weaker versions of strategy-proofness for SCCs that take into account the limitation of information at disposal of individuals. The classic definition of manipulability for resolute SCC, as well as its generalization to not necessarily resolute SCCs, requires that there is an individual who could potentially misreport her preferences based on the knowledge of others' reported preferences. Thus, the failure of strategy-proofness implies the existence of an individual who has the capability to precisely know others' preferences. That implicit assumption is definitely unrealistic in many contexts, as it is unlikely that anyone could access such detailed information. As a consequence, violating strategy-proofness may not always be a significant issue. On the other hand, Nurmi (1987), in his analysis of some classic SCCs, interestingly observes that an individual might decide to deviate based on a smaller amount of information about others' preferences. Thus, manipulability issues become much more significant if the information an individual needs for being profitable to misrepresent her preferences is small enough and easy to obtain. That suggests the possibility to consider notions of strategy-proofness where individual information about the other's preferences is incomplete and only limited to some specific features.

This approach has been developed for resolute SCCs in Conitzer et al. (2011), Reijngoud and Endriss (2012), and Gori (2021). Conitzer et al. (2011) associate with each individual a family of sets, called information sets, whose elements are lists of the others' preferences, and assume that the individual can identify an information set containing the true list of others' preferences; Reijngoud and Endriss (2012) assume instead that individuals are given only some pieces of information extracted by an opinion poll and described via a so-called poll information function; Gori (2021) generalized the two aforementioned frameworks by letting the information sets depend on individual preferences, as well. The general idea behind the concept of strategy-proofness with limited information is similar for all the described approaches: an individual is not fully aware of the others' preferences but she only knows that the partial preference profile built using the preferences of the others belongs to a specific set; an individual decides to report false preferences if, for every partial preference profile in

[1] An analysis of extension rules can be found in Barberà et al.(2004).

that set, false preferences cannot make her worse off and, for at least one partial preference profile in that set, they make her better off. Other contributions in the framework of resolute SCCs are due to Endriss et al. (2016) and Veselova (2020).[2]

In this paper we extend the approach by Gori (2021) to the framework of not necessarily resolute SCCs. We propose a general definition of manipulability for SCCs under incomplete information based on two fundamental parameters: an extension rule, which describes how to get a preference over sets of alternatives from any relation over alternatives, and a so-called information function profile, which describes the level of knowledge of each individual (Definition 4). After proposing a comparison between the general definition and other standard definitions of strategy-proofness, we focus on a particular but remarkable case. Specifically, we consider the well-known extension rule by Kelly (1977) and the so-called winner information function profile, which formalizes the fact that each individual, for a given SCC, is assumed to have the capability to anticipate the set of selected alternatives. Consequently, each individual knows that the partial preference profile formed by the others' preferences has the property that, if combined with her own preferences, it generates a preference profile that leads the SCC to determine a given outcome. Then, we carry out a detailed analysis of manipulability for three well-known SCCs, namely the Borda SCC, the plurality SCC, and the negative plurality SCC. We show that those SCCs exhibit significant differences among themselves and that their manipulability properties strongly depend on the number of individuals and the number of alternatives (Theorems 9, 10, and 11). We stress that the idea of winner information function profile was introduced and studied by Conitzer et al. (2011) in the framework of resolute SCCs, and then deepened in the same framework by other authors (Reijngoud and Endriss, 2012; Endriss et al. 2016, Veselova and Karabekyan, 2023). It is also worth mentioning that the analysis of strategy-proofness under incomplete information for multi-valued voting rule has been recently considered by Tsiaxiras (2021) in a framework different from the one of SCCs.

2 Preliminary results

Given $k \in \mathbb{N}$, we set $[\![k]\!] := \{x \in \mathbb{N} : x \leqslant k\}$.

Let X be a nonempty and finite set. We denote by $|X|$ the size of X; by $P(X)$ the set of the subsets of X; by $P_0(X)$ the set of the nonempty subsets of X; by $\mathrm{Sym}(X)$ the set of bijective functions from X to X. For $x, y \in X$, the bijection $\psi \in \mathrm{Sym}(X)$ such that $\psi(x) = y$, $\psi(y) = x$ and, for every $z \in X \setminus \{x, y\}$, $\psi(z) = z$, is called the transposition that exchanges x and y. A relation on X is a subset of X^2, that is, an element of $P(X^2)$. The set of relations on X is denoted by $\mathcal{R}(X)$.

Let $R \in \mathcal{R}(X)$. Given $x, y \in X$, we sometimes write $x \succeq_R y$ instead of $(x, y) \in R$; $x \succ_R y$ instead of $(x, y) \in R$ and $(y, x) \notin R$. Note that $x \succ_R y$ implies $x \succeq_R y$ and $x \neq y$. We say that x and y are R-comparable if at least one between $x \succeq_R y$ and $y \succeq_R x$ holds true.

We say that R is

- reflexive if, for every $x \in X$, $x \succeq_R x$;
- complete if, for every $x, y \in X$, $x \succeq_R y$ or $y \succeq_R x$;
- transitive if, for every $x, y, z \in X$, $x \succeq_R y$ and $y \succeq_R z$ imply $x \succeq_R z$;
- antisymmetric if, for every $x, y \in X$, $x \succeq_R y$ and $y \succeq_R x$ imply $x = y$;
- a partial order if R is reflexive, transitive and antisymmetric;
- a linear order on X if R is complete, transitive and antisymmetric.

Note that, if R is antisymmetric, then, for every $x, y \in X$, $x \succ_R y$ if and only if $x \succeq_R y$ and $x \neq y$; if $R' \in \mathcal{R}(X)$ is antisymmetric and $R \subseteq R'$, then, for every $x, y \in X$, $x \succ_R y$ implies $x \succ_{R'} y$.

[2] Terzopoulou and Endriss (2019) investigate the problem of manipulation under partial information in the framework of judgment aggregation.

For every $\psi \in \mathrm{Sym}(X)$, we set $\psi R = \{(x, y) \in X^2 : (\psi^{-1}(x), \psi^{-1}(y)) \in R\}$. Hence, for every $x, y \in X$, $x \succeq_R y$ if and only if $\psi(x) \succeq_{\psi R} \psi(y)$. We denote by $\mathcal{L}(X)$ the set of linear orders on X. Let $R \in \mathcal{L}(X)$ and $|X| = n$ with $n \in \mathbb{N}$. Then $\mathrm{rank}_R : X \to [\![n]\!]$ is the bijective function defined, for every $x \in X$, by $\mathrm{rank}_R(x) = |\{y \in X : y \succeq_R x\}|$.

For every $i \in [\![n]\!]$, let $x_i \in X$ be the unique element in X such that $\mathrm{rank}_R(x_i) = i$. Then R is completely determined by the ordered list $(x_i)_{i=1}^n \in X^n$ and thus we represent R by the writing $[x_1, \ldots, x_n]$.

3 Social choice correspondences

Let us fix two nonempty and finite sets A and I with $|A| \geqslant 2$ and $|I| \geqslant 2$. We interpret A as set of alternatives and I as set of individuals. For every $J \subseteq I$, we denote by $\mathcal{L}(A)^J$ the set of functions from J to $\mathcal{L}(A)$; the elements of $\mathcal{L}(A)^J$ are called preference profiles of individuals in J; any $p \in \mathcal{L}(A)^J$ represents a complete description of the preferences on A of the individuals in J by interpreting, for every $i \in J$, $p(i) \in \mathcal{L}(A)$ as the preferences on A of individual i. If, for a given $i \in J$, $p(i) = [x_1, \ldots, x_{|A|}]$, we refer to x_1 as the best alternative for individual i and to $x_{|A|}$ as the worst alternative for individual i.

The elements of $\mathcal{L}(A)^I$ are simply called preference profiles. In order to simplify the reading, in the rest of the paper the elements of $\mathcal{L}(A)^I$ will be usually denoted by p, possibly with suitable superscripts, and the elements of $\mathcal{L}(A)^{I \setminus \{i\}}$, where $i \in I$, will be usually denoted by $\overline{p}$, possibly with suitable superscripts. If $i \in I$ and $q \in \mathcal{L}(A)$, we denote by $q[i]$ the element of $\mathcal{L}(A)^{\{i\}}$ such that $q[i](i) = q$. Given $i \in I$, $\overline{p} \in \mathcal{L}(A)^{I \setminus \{i\}}$ and $q \in \mathcal{L}(A)$, the writing $(\overline{p}, q[i])$ represents the element of $\mathcal{L}(A)^I$ such that $(\overline{p}, q[i])(i) = q$ and $(\overline{p}, q[i])(j) = \overline{p}(j)$ for all $j \in I \setminus \{i\}$.

A social choice correspondence (SCC) is a function from $\mathcal{L}(A)^I$ to $P_0(A)$. Thus, a social choice correspondence is a procedure that associates with every preference profile a nonempty subset of A. A SCC F is resolute if, for every $p \in \mathcal{L}(A)^I$, $|F(p)| = 1$. For simplicity, if F is a resolute SCC, we identify $F(p)$ with the unique element of $F(p)$.

Let us recall now the definition of a very notable family of SCCs, namely the so-called positional SCCs. Consider a scoring vector, namely a vector $w = (w_1, \ldots, w_{|A|}) \in \mathbb{R}^{|A|}$ such that $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_{|A|}$ and $w_1 > w_{|A|}$. Given $p \in \mathcal{L}(A)^I$ and $x \in A$, the w-score of x at p is defined by

$$\mathrm{sc}_w(x, p) := \sum_{i \in I} w_{\mathrm{rank}_{p(i)}(x)}.$$

The positional SCC with scoring vector w, or briefly w-positional SCC, is the SCC that associates, with every $p \in \mathcal{L}(A)^I$, the set

$$\operatorname*{argmax}_{x \in A} \mathrm{sc}_w(x, p).$$

The Borda SCC, the plurality SCC, and the negative plurality SCC, respectively denoted by BO, PL and NP, are well-known positional SCCs respectively defined using the scoring vectors $w_{\mathrm{bo}} = (|A|-1, |A|-2, \ldots, 0)$, $w_{\mathrm{pl}} = (1, 0, \ldots, 0)$ and $w_{\mathrm{np}} = (1, 1, \ldots, 1, 0)$. The w_{bo}-score, the w_{pl}-score, and the w_{np}-score are respectively called the Borda score, the plurality score, and the negative plurality score and are simply denoted by bo, pl and np. Observe that, for every $p \in \mathcal{L}(A)^I$ and $x \in A$, we have

$$\mathrm{bo}(x, p) := \mathrm{sc}_{w_{\mathrm{bo}}}(x, p) = \sum_{i \in I} \big(|A| - \mathrm{rank}_{p(i)}(x)\big),$$

$$\mathrm{pl}(x, p) := \mathrm{sc}_{w_{\mathrm{pl}}}(x, p) = |\{i \in I : \mathrm{rank}_{p(i)}(x) = 1\}|,$$

$$\mathrm{np}(x, p) := \mathrm{sc}_{w_{\mathrm{np}}}(x, p) = |\{i \in I : \mathrm{rank}_{p(i)}(x) \neq |A|\}|.$$

It is easily checked that BO, PL and NP are not resolute unless $|A| = 2$ and $|I|$ is odd.

A SCC F is called unanimous if, for every $p \in \mathcal{L}(A)^I$ and $x \in A$, the fact that $\mathrm{rank}_{p(i)}(x) = 1$ for all $i \in I$ implies $F(p) = \{x\}$. It is a simple exercise to prove that a positional SCC with scoring vector w is unanimous if and only if $w_1 > w_2$.

4 Extension rules

An extension rule is a function $\mathbf{E}$ from $\mathcal{L}(A)$ to $\mathcal{R}(P_0(A))$ such that, for every $q \in \mathcal{L}(A)$ and $x, y \in A$, $\{x\} \succeq_{\mathbf{E}(q)} \{y\}$ if and only if $x \succeq_q y$ (Barberà et al., 2004). If $\mathbf{E}$ is an extension rule and $q \in \mathcal{L}(A)$ represents the preferences on A of an individual, we interpret the relation $\mathbf{E}(q)$ as a description of the preferences of that individual on the set $P_0(A)$. The problem of reasonably extending the preferences of an individual from a set of alternatives to the set of subsets of alternatives is crucial and largely investigated, and there are a variety of extension rules considered in the literature, each of them based on a specific rationale.

In this paper, we focus on the well-known Kelly extension rule (Kelly, 1977), defined, for every $q \in \mathcal{L}(A)$, as

$$\mathbf{K}(q) := \{(B, C) \in P_0(A)^2 : B = C\} \cup \left\{(B, C) \in P_0(A)^2 : x \succeq_q y \text{ for all } x \in B \text{ and } y \in C\right\}.$$

It is simple to prove that $\mathbf{K}$ is actually an extension rule. Moreover, for every $q \in \mathcal{L}(A)$, $\mathbf{K}(q)$ is a partial order on $P_0(A)$ that, in general, is far from being complete. Proposition 18 in Appendix A collects some basic properties of $\mathbf{K}$ that will be used throughout the paper without reference.

5 Manipulation of SCCs

Let us recall the fundamental definition of strategy-proofness for resolute SCC.

Definition 1. *Let F be a resolute* SCC*. We say that F is manipulable if there exist $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$ such that $F(\overline{p}, q'[i]) \succ_q F(\overline{p}, q[i])$. We say that F is strategy-proof if it is not manipulable.*

The following definition, which corresponds to Definition 2 in Brandt and Brill (2011), extends the standard concept of strategy-proofness, originally introduced for resolute SCCs, to the case of SCCs that are not necessarily resolute.

Definition 2. *Let F be a* SCC *and $\mathbf{E}$ be an extension rule. We say that F is $\mathbf{E}$-manipulable if there exist $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$ such that $F(\overline{p}, q'[i]) \succ_{\mathbf{E}(q)} F(\overline{p}, q[i])$. We say that F is $\mathbf{E}$-strategy-proof if it is not $\mathbf{E}$-manipulable.*

Note that if F is a resolute SCC and $\mathbf{E}$ an extension rule, then F is $\mathbf{E}$-manipulable if and only if F is manipulable.

In recent years, some authors have started considering the problem of manipulations of resolute social choice correspondences under the assumptions that individuals are not able to exactly know the preferences of the others, as assumed in Definitions 1 and 2, but only have limited information about them (Conitzer et al., 2011, Reijngoud and Endriss, 2012, Endriss et al., 2016, Veselova, 2020, Gori, 2021). An effective way to model the different levels of information of individuals is the use of the so-called information function profiles, which are introduced in Gori (2021), and generalize both the concept of information sets by Conitzer et al. (2011) and the one of poll information function by Reijngoud and Endriss (2012).

Let $i \in I$. An information function for individual i is a function $\Omega_i : \mathcal{L}(A) \to P_0(P_0(\mathcal{L}(A)^{I\setminus\{i\}}))$. Thus, Ω_i associates a nonempty set of nonempty subsets of $\mathcal{L}(A)^{I\setminus\{i\}}$ with each $q \in \mathcal{L}(A)$. The idea is that if individual i has preference q, then the type of information that she has about the preferences of the others realizes in her capability to identify a set $\omega \in \Omega_i(q)$ which surely contains among its elements the preference profile of the other individuals. An information function profile is a list $\Omega = (\Omega_i)_{i\in I}$ that collects the information functions of all the individuals. Let us now present three basic examples of information function profiles, which have been first introduced and investigated by Conitzer et al. (2011).

The *complete information function profile*, denoted by Ω^c, is defined, for every $i \in I$ and $q \in \mathcal{L}(A)$, by

$$\Omega_i^c(q) := \left\{\{\overline{p}\} : \overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}\right\}.$$

When this information function profile is considered, every $\omega \in \Omega_i^c(q)$ consists of exactly one preference profile of individuals in $I\backslash\{i\}$. Thus, Ω^c describes the situation where each individual exactly knows the preferences of the other individuals.

The *zero information function profile*, denoted by Ω^0, is defined, for every $i \in I$ and $q \in \mathcal{L}(A)$, by

$$\Omega_i^0(q) \coloneqq \left\{\mathcal{L}(A)^{I\backslash\{i\}}\right\}.$$

In this case, every $\omega \in \Omega_i^0(q)$ consists of the whole set $\mathcal{L}(A)^{I\backslash\{i\}}$. Thus, Ω^0 describes the situation where each individual only knows the obvious fact that the preference profile of individuals in $I\backslash\{i\}$ belongs to $\mathcal{L}(A)^{I\backslash\{i\}}$.

Note that, for every $q, q' \in \mathcal{L}(A)$, $\Omega^c(q) = \Omega^c(q')$ and $\Omega^0(q) = \Omega^0(q')$. Thus, for those two information function profiles the dependence on q is fictitious. We now introduce a significant example where the dependence of q does play a role.

Given a SCC F, the *F-winner information function profile*, denoted by Ω^F, is defined, for every $i \in I$ and $q \in \mathcal{L}(A)$, by

$$\Omega_i^F(q) \coloneqq \left\{\{\overline{p} \in \mathcal{L}(A)^{I\backslash\{i\}} : F(\overline{p}, q[i]) = X\} : X \in \mathrm{Im}(F)\right\}\backslash\{\varnothing\}.$$

In this case, each $\omega \in \Omega_i^F(q)$ is formed by all the preference profiles that, completed with q as the preference of individual i, give the same outcome. Thus, Ω^F describes the situation where the information that each individual has is the knowledge of the final outcome obtained applying F to the preferences of the individuals in the society. Note that Ω^F is actually an information function profile, that is, for every $i \in I$ and $q \in \mathcal{L}(A)$, $\Omega_i^F(q) \neq \varnothing$. Indeed, pick $\overline{p}^* \in \mathcal{L}(A)^{I\backslash\{i\}}$ and let $X^* = F(\overline{p}^*, q[i])$. Then $X^* \in \mathrm{Im}(F)$ and $\overline{p}^* \in \{\overline{p} \in \mathcal{L}(A)^{I\backslash\{i\}} : F(\overline{p}^*, q[i]) = X^*\} \neq \varnothing$. Thus, $\{\overline{p} \in \mathcal{L}(A)^{I\backslash\{i\}} : F(\overline{p}^*, q[i]) = X^*\} \in \Omega_i^F(q) \neq \varnothing$.

The following definition, which corresponds to Definition 3 in Gori (2021), describes the meaning of manipulability and strategy-proofness for resolute SCCs when individuals' information about the preferences of the others is described by a suitable information function profile.

Definition 3. *Let F be a resolute* SCC *and Ω be an information function profile. We say that F is Ω-manipulable if there exist $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\omega \in \Omega_i(q)$ such that*

- *there exists $\overline{p}' \in \omega$ with $F(\overline{p}', q'[i]) \succ_q F(\overline{p}', q[i])$;*
- *for every $\overline{p} \in \omega$, we have $F(\overline{p}, q'[i]) \succeq_q F(\overline{p}, q[i])$.*

We say that F is Ω-strategy-proof if it is not Ω-manipulable.

By combining Definitions 2 and 3 it is possible to introduce the concepts of manipulability and strategy-proofness for SCCs that are not necessarily resolute, under the assumption of incomplete information.

Definition 4. *Let F be a* SCC*, $\mathbf{E}$ be an extension rule and Ω be an information function profile. We say that F is Ω-$\mathbf{E}$-manipulable if there exist $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\omega \in \Omega_i(q)$ such that*

- *there exists $\overline{p}' \in \omega$ with $F(\overline{p}', q'[i]) \succ_{\mathbf{E}(q)} F(\overline{p}', q[i])$;*
- *for every $\overline{p} \in \omega$, we have $F(\overline{p}, q'[i]) \succeq_{\mathbf{E}(q)} F(\overline{p}, q[i])$.*

We say that F is Ω-$\mathbf{E}$-strategy-proof if it is not Ω-$\mathbf{E}$-manipulable.

Thus, F is Ω-$\mathbf{E}$-strategy proof if, every time an individual i, whose preferences are described by $q \in \mathcal{L}(A)$, knows that the preferences of the others are surely described by some element of $\omega \in \Omega_i(q)$ and observes that for an element $\overline{p}' \in \omega$ it is convenient for her to report the false preferences described by q', then there exists another element $\overline{p} \in \omega$ for which $F(\overline{p}, q'[i]) \not\succeq_{\mathbf{E}(q)} F(\overline{p}, q[i])$, that is, one of the two following situation holds true:

- $F(\overline{p}, q\,[i]) \succ_{\mathbf{E}(q)} F(\overline{p}, q'\,[i])$, meaning that, for the element $\overline{p} \in \omega$, she would be better off if she told the truth;
- $F(\overline{p}, q\,[i])$ and $F(\overline{p}, q'\,[i])$ are not $\mathbf{E}(q)$-comparable, meaning that, for the element $\overline{p} \in \omega$, she is not able to compare the two sets $F(\overline{p}, q\,[i])$ and $F(\overline{p}, q'\,[i])$.

Note that

$$F \text{ is } \Omega^c\text{-}\mathbf{E}\text{-manipulable if and only if } F \text{ is } \mathbf{E}\text{-manipulable.} \tag{1}$$

Moreover, if F is resolute, F is Ω-$\mathbf{E}$-manipulable if and only if F is Ω-manipulable, and F is Ω^c-$\mathbf{E}$-manipulable if and only if F is manipulable.

We end this section with some propositions that allow to deduce the Ω'-$\mathbf{E}'$-manipulability of a social choice correspondence from its Ω-$\mathbf{E}$-manipulability, provided that suitable properties of the information function profiles Ω and Ω' and the extension rules $\mathbf{E}$ and $\mathbf{E}'$ are satisfied.

Given two extension rules $\mathbf{E}$ and $\mathbf{E}'$, we say that $\mathbf{E}$ is a refinement of $\mathbf{E}'$ if, for every $q \in \mathcal{L}(A)$, $\mathbf{E}(q) \subseteq \mathbf{E}'(q)$; if $\mathbf{E}$ is a refinement of $\mathbf{E}'$, we write $\mathbf{E} \subseteq \mathbf{E}'$.

Proposition 5. *Let F be a* SCC*, $\mathbf{E}$ and $\mathbf{E}'$ be extension rules with $\mathbf{E} \subseteq \mathbf{E}'$, and Ω be an information function profile. Assume that F is Ω-$\mathbf{E}$-manipulable and that, for every $q \in \mathcal{L}(A)$, $\mathbf{E}'(q)$ is antisymmetric. Then F is Ω-$\mathbf{E}'$-manipulable.*

Proof. Since F is Ω-$\mathbf{E}$-manipulable, we have that there exist $i \in I$, $q, q' \in \mathcal{L}(A)$, $\omega \in \Omega_i(q)$ and $\overline{p}' \in \omega$ such that $F(\overline{p}', q'\,[i]) \succ_{\mathbf{E}(q)} F(\overline{p}', q\,[i])$; for every $\overline{p} \in \omega$, we have $F(\overline{p}, q'\,[i]) \succeq_{\mathbf{E}(q)} F(\overline{p}, q\,[i])$. Since $\mathbf{E} \subseteq \mathbf{E}'$ we immediately have that, for every $\overline{p} \in \omega$, $F(\overline{p}, q'\,[i]) \succeq_{\mathbf{E}'(q)} F(\overline{p}, q\,[i])$. Since $F(\overline{p}', q'\,[i]) \succ_{\mathbf{E}(q)} F(\overline{p}', q\,[i])$ and $\mathbf{E}'(q)$ is antisymmetric, we have $F(\overline{p}', q'\,[i]) \succ_{\mathbf{E}'(q)} F(\overline{p}', q\,[i])$. Then, F is Ω-$\mathbf{E}'$-manipulable. □

Given Ω and Ω' information function profiles, we say that Ω is at least as informative as Ω', and we write $\Omega \trianglerighteq \Omega'$, if, for every $i \in I$, $q \in \mathcal{L}(A)$ and $\omega' \in \Omega'_i(q)$, there exists $\mathcal{A} \subseteq \Omega_i(q)$ with $\mathcal{A} \neq \varnothing$ such that $\omega' = \bigcup_{\omega \in \mathcal{A}} \omega$ (Gori, 2021, Definition 4). Note that, for every information function profile Ω, we have $\Omega^c \trianglerighteq \Omega$.

Proposition 6. *Let F be a* SCC*, $\mathbf{E}$ be an extension rule, and Ω and Ω' be information function profiles such that $\Omega \trianglerighteq \Omega'$. Assume that F is Ω'-$\mathbf{E}$-manipulable. Then F is Ω-$\mathbf{E}$-manipulable.*

Proof. Since F is Ω'-$\mathbf{E}$-manipulable, then there exist $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\omega' \in \Omega'_i(q)$ such that, for every $\overline{p} \in \omega'$, we have $F(\overline{p}, q'\,[i]) \succeq_{\mathbf{E}(q)} F(\overline{p}, q\,[i])$, and there exists $\overline{p}' \in \omega'$ such that $F(\overline{p}', q'\,[i]) \succ_{\mathbf{E}(q)} F(\overline{p}', q\,[i])$. Since $\Omega \trianglerighteq \Omega'$, we have $\omega' = \bigcup_{\omega \in \mathcal{A}} \omega$, for some $\mathcal{A} \subseteq \Omega_i(q)$ with $\mathcal{A} \neq \varnothing$. Then there exists $\omega^* \in \mathcal{A}$ such that $\overline{p}' \in \omega^*$. Since $\omega^* \subseteq \omega'$ we have that, for every $\overline{p} \in \omega^*$, $F(\overline{p}, q'\,[i]) \succeq_{\mathbf{E}(q)} F(\overline{p}, q\,[i])$. Since we know that $F(\overline{p}', q'\,[i]) \succ_{\mathbf{E}(q)} F(\overline{p}', q\,[i])$ and $\overline{p}' \in \omega^*$, we conclude that F is Ω-$\mathbf{E}$-manipulable. □

It is also worth mentioning the two following corollaries that are immediate consequences of (1), Propositions 5 and 6, and the properties of Ω^c.

Corollary 7. *Let F be a* SCC*, and $\mathbf{E}$ and $\mathbf{E}'$ be extension rules with $\mathbf{E} \subseteq \mathbf{E}'$. Assume that F is $\mathbf{E}$-manipulable and that, for every $q \in \mathcal{L}(A)$, $\mathbf{E}'(q)$ is antisymmetric. Then F is $\mathbf{E}'$-manipulable.*

Corollary 8. *Let F be a* SCC*, $\mathbf{E}$ an extension rule and Ω an information function profile. Assume that F is Ω-$\mathbf{E}$-manipulable. Then F is $\mathbf{E}$-manipulable.*

6 Main results

Suppose that one is interested in using a certain SCC F, and suppose that the extension rule $\mathbf{E}$ carefully describes the way individuals extend their preferences from A to $P_0(A)$. Given an information function profile Ω, one and only one of the following three situations can occur

- F is $\mathbf{E}$-strategy-proof (and then Ω-$\mathbf{E}$-strategy-proof),

- F is **E**-manipulable and Ω-**E**-strategy-proof,
- F is Ω-**E**-manipulable (and then **E**-manipulable),

and it is certainly interesting to understand which one actually occurs. Indeed, if F is **E**-strategy-proof, then we know that no possible strategic behavior may be implemented; if F is **E**-manipulable and Ω-**E**-strategy-proof, then we know that full information may cause a possible deviation but if the level of information of individuals corresponds at most to the one described by Ω nobody has an incentive to report false preferences; if F is Ω-**E**-manipulable, then individuals have an incentive to report false preferences even if their level of information corresponds at least to the one described by Ω.

In this paper, we focus on the Kelly extension rule and the winner information function profile, and carry on the aforementioned analysis for some well-known SCCs. In particular, we develop a full-fledged analysis for the Borda SCC, the plurality SCC, and the negative plurality SCC. The results obtained are summarized by the following three theorems.

Theorem 9. *If $|A| = 2$, then BO is **K**-strategy-proof. If $|A| \geqslant 3$, then BO is Ω^{BO}-**K**-manipulable.*

Theorem 10. *If $|A| = 2$, then PL is **K**-strategy-proof. If $|A| \geqslant 3$ and $|I| \in \{2, 3\}$, then PL is **K**-strategy-proof. If $|A| \geqslant 3$ and $|I| \geqslant 4$, then PL is Ω^{PL}-**K**-manipulable.*

Theorem 11. *If $|A| = 2$, then NP is **K**-strategy-proof. If $|A| = 3$ and 3 divides $|I| - 1$, then NP is **K**-manipulable and Ω^{NP}-**K**-strategy-proof. If $|A| = 3$ and 3 does not divide $|I| - 1$, then NP is Ω^{NP}-**K**-manipulable. If $|A| \geqslant 4$ and $|I| < |A| - 1$, then NP is **K**-strategy-proof. If $|A| \geqslant 4$ and $|I| \geqslant |A| - 1$, then NP is **K**-manipulable and Ω^{NP}-**K**-strategy-proof.*

The aforementioned theorems state that, when the alternatives are two, BO, PL and NP are **K**-strategy-proof. That fact is indeed true for any positional SCC as shown in Section 7. If the alternatives are at least three, some important differences among the three SCCs start emerging. Indeed, BO is always Ω^{BO}-**K**-manipulable; PL is Ω^{BO}-**K**-manipulable unless the number of individuals is two or three; NP exhibits instead a much more complex behavior. It is worth noting that, depending on the arithmetical relation between $|A|$ and $|I|$, all the three possible scenarios can occur for NP. In particular, if $|I| \geqslant |A| - 1$ and the individuals have a level of information corresponding at most to the knowledge of the winners, then no individual can manipulate NP in the sense of Kelly.

The proofs of Theorems 9, 10 and 11, which are given in Appendixes B and C, are partially based on further results presented in Sections 7 and 8. More exactly, in Section 7 we consider the case where the alternatives are two and we prove that the qualified majority is **K**-strategy-proof (Proposition 13); in Section 8 we provide conditions that, if satisfied by a SCC F, guarantee that F is Ω^F-**K**-manipulable (Theorem 16). By means of that conditions, we also deduce the following result that clearly implies the statement related to the case where $|A| \geqslant 3$ and $|I| \geqslant 4$ in Theorems 9 and 10. The proof of Theorem 12 is in Appendix B.

Theorem 12. *Assume that $|A| \geqslant 3$ and $|I| \geqslant 4$. If F is an unanimous positional* SCC*, then F is Ω^F-**K**-manipulable.*

The analysis of positional rules when $|I| \in \{2, 3\}$ or when they are not unanimous is more difficult and seems to lead to a variety of different situations, as also emerges from Theorems 9, 10 and 11. We stress that Theorem 12 is analogous to Theorem 3 in Reijngoud and Endriss (2012), where the authors focus on unanimous and positional SCCs made resolute by an agenda for breaking ties. Also the proofs of those theorems share some similarities. However, none of the two is an immediate corollary of the other.

7 The case $|A| = 2$

Assume that $A = \{a, b\}$ with $a \neq b$. Let $\alpha \geqslant \frac{|I|}{2}$. The α-majority SCC, here denoted by MAJ_α, is defined, for every $p \in \mathcal{L}(A)^I$, by

$$MAJ_\alpha(p) \coloneqq \begin{cases} \{a\} & \text{if } |\{i \in I : a \succeq_{p(i)} b\}| > \alpha \\ \{b\} & \text{if } |\{i \in I : b \succeq_{p(i)} a\}| > \alpha \\ \{a, b\} & \text{otherwise} \end{cases}$$

It is easily observed that each positional SCC on two alternatives coincides with $MAJ \coloneqq MAJ_{\frac{|I|}{2}}$.

Proposition 13. *Assume that $|A| = 2$ and $\alpha \geqslant \frac{|I|}{2}$. Then MAJ_α is $\mathbf{K}$-strategy-proof.*

Proof. Let $i \in I$, $\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$ and $q, q' \in \mathcal{L}(A)$. Let us set $p = (\overline{p}, q[i])$ and $p' = (\overline{p}, q'[i])$ and assume that $q = [x_1, x_2]$, where $A = \{x_1, x_2\}$. We want to prove that

$$MAJ_\alpha(p') \not\succ_{\mathbf{K}(q)} MAJ_\alpha(p). \tag{2}$$

If $q' = q$, (2) is true. Assume $q' \neq q$ and so $q' = [x_2, x_1]$. If $MAJ_\alpha(p) = \{x_1\}$, we have that $MAJ_\alpha(p) \succeq_{\mathbf{K}(q)} B$ for all $B \in P_0(A)$ and thus (2) holds true. Assume next that $MAJ_\alpha(p) \neq \{x_1\}$. Thus, $x_2 \in MAJ_\alpha(p)$ and $|\{i \in I : x_1 \succeq_{p(i)} x_2\}| \leqslant \alpha$. Then, we have that $|\{i \in I : x_1 \succeq_{p'(i)} x_2\}| < |\{i \in I : x_1 \succeq_{p(i)} x_2\}| \leqslant \alpha$. As a consequence, $x_2 \in MAJ_\alpha(p')$. If $x_1 \in MAJ_\alpha(p)$, then (2) is true. If $x_1 \notin MAJ_\alpha(p)$, then $MAJ_\alpha(p) = \{x_2\}$. Thus, $|\{i \in I : x_2 \succeq_{p'(i)} x_1\}| > |\{i \in I : x_2 \succeq_{p(i)} x_1\}| > \alpha$ and hence $MAJ_\alpha(p') = \{x_2\}$. As a consequence, (2) is true. □

8 Sufficient conditions for Ω^F-K-manipulability

In this section, we propose a result that gives conditions on a SCC F that are sufficient for its Ω^F-$\mathbf{K}$-manipulability. We start by defining a property of monotonicity. Similar properties are proposed in the literature, as in Chapter 4 of Campbell et al. (2018).

Definition 14. *Let F be a SCC. We say that F satisfies up-monotonicity (UM) if, for every $i \in I$, $\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$, $q \in \mathcal{L}(A)$ and $x, y, z \in A$ such that*

- $F(\overline{p}, q[i]) = \{z\}$,
- $\mathrm{rank}_q(x) + 1 = \mathrm{rank}_q(y) < \mathrm{rank}_q(z)$,

we have that $F(\overline{p}, \psi q\,[i]) \subseteq \{z, y\}$, where $\psi \in \mathrm{Sym}(A)$ is the transposition that exchanges x and y.

A SCC satisfies UM if, for every preference profile for which exactly one winner is selected, if an individual exchanges in her preferences the position of two consecutive alternatives that are ranked above the winner, then one of the following facts happens: the set of winners does not change; the raised alternative becomes the unique winner; the raised alternative becomes a winner together with the former winner. Note that if $|A| = 2$, then every SCC satisfies UM.

Definition 15. *Let F be a SCC. We say that F satisfies up-sensitivity (US) if there exist $i \in I$, $\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$, $q \in \mathcal{L}(A)$ and $x, y, z \in A$ such that*

- $F(\overline{p}, q[i]) = \{z\}$,
- $\mathrm{rank}_q(x) + 1 = \mathrm{rank}_q(y) < \mathrm{rank}_q(z)$,
- *$y \in F(\overline{p}, \psi q\,[i])$, where $\psi \in \mathrm{Sym}(A)$ is the transposition that exchanges x and y.*

A SCC satisfies US if there exists a preference profile for which exactly one winner is selected, and an individual who, by exchanging the position of two consecutive alternatives that are not winners and are ranked above the winner in her preferences, makes the set of winners contain the raised alternative. Note that if $|A| = 2$, then no SCC satisfies US.

We now state and prove the main result of the section.

Theorem 16. *Let F be a* SCC. *Assume that F satisfies* UM *and* US. *Then F is Ω^F-$\mathbf{K}$-manipulable.*

Proof. In order to prove that F is Ω^F-$\mathbf{K}$-manipulable, we need to show that there exist $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\omega \in \Omega_i^F(q)$ such that, for every $\overline{p} \in \omega$, we have $F(\overline{p}, q'[i]) \succeq_{\mathbf{K}(q)} F(\overline{p}, q[i])$ and there exists $\overline{p}' \in \omega$ such that $F(\overline{p}', q'[i]) \succ_{\mathbf{K}(q)} F(\overline{p}', q[i])$.

Since F satisfies US there exist $i \in I$, $\overline{p}' \in \mathcal{L}(A)^{I\setminus\{i\}}$, $q \in \mathcal{L}(A)$ and $x, y, z \in A$ such that

- $F(\overline{p}', q[i]) = \{z\}$,
- $\mathrm{rank}_q(x) + 1 = \mathrm{rank}_q(y) < \mathrm{rank}_q(z)$,
- $y \in F(\overline{p}', \psi q[i])$, where ψ is the transposition that exchanges x and y.

In particular, since F satisfies UM, we have that $F(\overline{p}', \psi q[i]) = \{y, z\}$ or $F(\overline{p}', \psi q[i]) = \{y\}$.

Let now $i \in I$ and $q \in \mathcal{L}(A)$ be the ones previously considered, and set $q' = \psi q$ and $\omega = \{\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}} : F(\overline{p}, q[i]) = \{z\}\}$. Since $F(\overline{p}', q[i]) = \{z\}$, we have that $\omega \neq \varnothing$ and so $\omega \in \Omega_i^F(q)$. Of course, we have that $F(\overline{p}', q'[i]) \succ_{\mathbf{K}(q)} \{z\} = F(\overline{p}', q[i])$, because $y \succ_q z$. We are then left with proving that, for every $\overline{p} \in \omega$, we have $F(\overline{p}, q'[i]) \succeq_{\mathbf{K}(q)} F(\overline{p}, q[i])$. Let $\overline{p} \in \omega$. Thus, we have $F(\overline{p}, q[i]) = \{z\}$. Moreover, by UM, we have $F(\overline{p}, \psi q[i]) = F(\overline{p}, q'[i]) \in \{\{z\}, \{y\}, \{y, z\}\}$. Since, for every $B \in \{\{z\}, \{y\}, \{y, z\}\}$ we have $B \succeq_{\mathbf{K}(q)} \{z\}$, we conclude that $F(\overline{p}, q'[i]) \succeq_{\mathbf{K}(q)} F(\overline{p}, q[i])$. □

Note that if $|A| = 2$, then no SCC can satisfy both UM and US. Thus, Theorem 16 is not informative if $|A| = 2$. As we will see by the proofs in the appendix, it is instead a useful tool to analyze positional SCCs when $|A| \geqslant 3$.

9 Conclusion

After having introduced the definition of Ω-$\mathbf{E}$-manipulability (Definition 4), we have focused on Ω^F-$\mathbf{K}$-manipulability. We have then provided sufficient conditions for Ω^F-$\mathbf{K}$-manipulability (Theorem 16) and analyzed Ω^F-$\mathbf{K}$-manipulability when F is positional (Theorems 9, 10, 11, and 12). The results for BO and PL confirm that it is very difficult to avoid manipulation for those SCCs, even when the information at the disposal of the individuals is significantly limited. On the other hand, the results for NP show that, for this SCC, limiting the information can be an effective way to achieve strategy-proofness.

This study can be extended and deepened by considering different extension rules, different information function profiles, and different families of social choice correspondences. Since different information function profiles may be comparable, meaning that one may be more or less informative than another, finding different results in terms of manipulability for different information function profiles might allow us to shed light on how much information is needed to manipulate a given SCC. By varying the extension rules, we modify our assumptions about how individuals evaluate sets of alternatives based on their preferences on alternatives. Of course, changing extension rules and information function profiles generally has a decisive impact on the definition of manipulability.

As an example, consider the well-known Fishburn extension rule (Gänderfors, 1976) defined, for every $q \in \mathcal{L}(A)$, by

$$\mathbf{F}(q) := \left\{(B, C) \in P_0(A)^2 : x \succeq_q y \ \text{ for all } x \in B\setminus C \text{ and } y \in C\right\}$$
$$\cap \left\{(B, C) \in P_0(A)^2 : x \succeq_q y \text{ for all } x \in B \text{ and } y \in C\setminus B\right\}.$$

Then the following result holds true.

Proposition 17. *If $|A| \geqslant 3$ and $|A|$ does not divide $|I| - 1$, then NP is Ω^{NP}-$\mathbf{F}$-manipulable.*

Proof. Let $|A| = n$ and $|I| = m$ and assume that $n \geqslant 3$, $m \geqslant 2$, and n does not divide $m - 1$. Assume also that $A = \{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are distinct. Let $i \in I$, $q = [x_1, \ldots, x_n]$ and $\omega = \{\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}} : NP(\overline{p}, q[i]) = A\setminus\{x_n\}\}$. Note that $\omega \in \Omega_i^F(q)$. Indeed, let $\overline{p}' \in \mathcal{L}(A)^{I\setminus\{i\}}$ be such that $\overline{p}'(i) = q$ for all $i \in I\setminus\{i\}$. We have that $\overline{p}' \in \omega \neq \varnothing$ and so $\omega \in \Omega_i^F(q)$.

Let ψ be the transposition that exchanges x_n and x_{n-1}. We show that NP is Ω^{NP}-**F**-manipulable by proving that, for every $\overline{p} \in \omega$, $NP(\overline{p}, \psi q[i]) = A \backslash \{x_{n-1}, x_n\}$. Indeed, since $N(\overline{p}, q[i]) = A \backslash \{x_n\}$, we deduce that, for every $\overline{p} \in \omega$, $NP(\overline{p}, \psi q[i]) \succ_{\mathbf{F}(q)} N(\overline{p}, q[i])$, which clearly implies Ω^{NP}-**F**-manipulability.

Fix $\overline{p} \in \omega$ and set $p = (\overline{p}, q[i])$. For every $x \in A$, set $N(x) \coloneqq \mathrm{np}(x, p)$, $N'(x) \coloneqq \mathrm{np}(x, (\overline{p}, \psi q[i]))$, and $L(x) \coloneqq |\{i \in I : \mathrm{rank}_{p(i)}(x) = n\}|$. We have that, for every $x \in A$, $L(x) = m - N(x)$. Moreover, $\sum_{x \in A} L(x) = m$ and, for every $x, y \in A \backslash \{x_n\}$, $L(x_n) > L(x) = L(y)$. Assume now by contradiction that $L(x_n) = L(x_{n-1}) + 1$. Then, we get

$$m = \sum_{x \in A} L(x) = L(x_n) + \sum_{x \in A \backslash \{x_n\}} L(x) = L(x_{n-1}) + 1 + (n-1)L(x_{n-1}) = nL(x_{n-1}) + 1.$$

Thus n divides $m-1$, a contradiction. Hence, $L(x_n) \geqslant L(x_{n-1}) + 2$ and so $N(x_n) \leqslant N(x_{n-1}) - 2$. We have $N'(x_n) = N(x_n) + 1$, $N'(x_{n-1}) = N(x_{n-1}) - 1$ and, for every $x \in A \backslash \{x_{n-1}, x_n\}$, $N'(x) = N(x)$. Note that, since $n \geqslant 3$, $A \backslash \{x_{n-1}, x_n\} \neq \varnothing$. Then, $NP(\overline{p}, \psi q[i]) = A \backslash \{x_{n-1}, x_n\}$, as desired. □

Proposition 17 shows that, the use of the Fishburn extension rule instead of Kelly extension rule, makes NP have a different behavior in terms of manipulability. Indeed, if $|A| \geqslant 4$ and $|A|$ does not divide $|I| - 1$, then, by Theorem 11 and Proposition 17, we have that NP is Ω^{NP}-**K**-strategy-proof and Ω^{NP}-**F**-manipulable.

Declarations of competing interest

The authors have no competing interests to declare that are relevant to the content of this article.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors thank the participants of the XLVIII AMASES Conference, for their comments. Raffaele Berzi and Michele Gori have been supported by local funding from the Università degli Studi di Firenze. Daniela Bubboloni has been supported by GNSAGA of INdAM (Italy), by the European Union - Next Generation EU, Missione 4 Componente 1, PRIN 2022-2022PSTWLB - Group Theory and Applications, CUP B53D23009410006, and by local funding from the Università degli Studi di Firenze.

Appendix

This appendix is devoted to the proofs of Theorems 9, 10, 11 and 12. Those proofs are developed through some intermediate steps. For simplicity, in the rest of the paper we set $|A| = n$ and $|I| = m$ and, without loss of generality, we assume that $A = [\![n]\!]$ and $I = [\![m]\!]$.

A Properties of the Kelly extension rule

Recall that, for every $q \in \mathcal{L}(A)$, $\mathbf{K}(q)$ is a partial order. The following proposition collects some further basic properties of the Kelly extension rule.

Proposition 18. *Let $q \in \mathcal{L}(A)$ with $q = [x_1, \ldots, x_{|A|}]$. Then, for every $B, C \in P_0(A)$, the following facts hold true.*

(*i*) *If $B \succ_{\mathbf{K}(q)} C$, then $|B \cap C| \leqslant 1$.*

(ii) *If $B \subsetneq C$ and B and C are $\mathbf{K}(q)$-comparable, then B is a singleton.*

(iii) *$C \not\succ_{\mathbf{K}(q)} B$ if and only if $B = C$ or there exist $x \in B$ and $y \in C$ such that $x \succ_q y$.*

(iv) *If $\mathrm{rank}_q(y) > \mathrm{rank}_q(x)$, then $\{x\} \succ_{\mathbf{K}(q)} \{x, y\} \succ_{\mathbf{K}(q)} \{y\}$.*

(v) *If $B \neq \{x_1\}$, then $\{x_1\} \succ_{\mathbf{K}(q)} B$.*

(vi) *If $x_1 \in B$ and $x_1 \notin C$, then $C \not\succ_{\mathbf{K}(q)} B$.*

Proof. *(i)* Let $B \succ_{\mathbf{K}(q)} C$. Suppose by contradiction that $|B \cap C| \geqslant 2$. Thus, there are $x, y \in B \cap C$ with $x \neq y$. Since $B \succeq_{\mathbf{K}(q)} C$ and $B \neq C$, we deduce that $x \succeq_q y$ and $y \succeq_q x$. By antisymmetry of q, we then get the contradiction $x = y$.

(ii) Let $B \subsetneq C$ and suppose that $B \succeq_{\mathbf{K}(q)} C$ or $C \succeq_{\mathbf{K}(q)} B$. Since $B \neq C$, we have $B \succ_{\mathbf{K}(q)} C$ or $C \succ_{\mathbf{K}(q)} B$. Thus, by *(i)*, we get $|B \cap C| \leqslant 1$. Since $B \cap C = B$, we have $B \cap C \neq \varnothing$. As a consequence, we deduce $|B| = |B \cap C| = 1$.

(iii) Let $C \not\succ_{\mathbf{K}(q)} B$ and $B \neq C$. Assume, by contradiction, that, for every $x \in B$ and $y \in C$, $x \not\succ_q y$. Since q is complete, we deduce that, for every $x \in B$ and $y \in C$, $y \succeq_q x$. Thus, $C \succeq_{\mathbf{K}(q)} B$. Since $\mathbf{K}(q)$ is antisymmetric and $B \neq C$, we finally get the contradiction $C \succ_{\mathbf{K}(q)} B$.

Assume, conversely, that $B = C$ or that there exist $x \in B$ and $y \in C$ such that $x \succ_q y$. If $B = C$, we immediately have $C \not\succ_{\mathbf{K}(q)} B$. If $B \neq C$, there must exist $x \in B$ and $y \in C$ such that $x \succ_q y$. Then, $y \not\succeq_q x$ and therefore $C \not\succeq_{\mathbf{K}(q)} B$. As a consequence, we also have $C \not\succ_{\mathbf{K}(q)} B$.

(iv)-*(vi)* Straightforward. □

B Proofs of Theorems 9, 10 and 12

Proposition 19. *Let F be a positional* SCC. *Then F satisfies* UM.

Proof. Assume that $F = PS_w$, where w is a suitable scoring vector. Consider $i \in I$, $\overline{p} \in \mathcal{L}(A)^{I \setminus \{i\}}$, $q \in \mathcal{L}(A)$ and $x, y, z \in A$ such that

- $F(\overline{p}, q[i]) = \{z\}$,
- $\mathrm{rank}_q(x) + 1 = \mathrm{rank}_q(y) < \mathrm{rank}_q(z)$,

and let $\psi \in \mathrm{Sym}(A)$ be the transposition that exchanges x and y.

We know that, for every $u \in A \setminus \{z\}$, $\mathrm{sc}_w(z, (\overline{p}, q[i])) > \mathrm{sc}_w(u, (\overline{p}, q[i]))$. It is immediately observed that $\mathrm{sc}_w(u, (\overline{p}, \psi q[i])) = \mathrm{sc}_w(u, (\overline{p}, q[i]))$ for all $u \in A \setminus \{x, y\}$. In particular, for every $u \in A \setminus \{x, y, z\}$, we have

$$\mathrm{sc}_w(z, (\overline{p}, \psi q[i])) = \mathrm{sc}_w(z, (\overline{p}, q[i])) > \mathrm{sc}_w(u, (\overline{p}, q[i])) = \mathrm{sc}_w(u, (\overline{p}, \psi q[i])).$$

Moreover, we have

$$\mathrm{sc}_w(x, (\overline{p}, \psi q[i])) \leqslant \mathrm{sc}_w(x, (\overline{p}, q[i])) < \mathrm{sc}_w(z, (\overline{p}, q[i])) = \mathrm{sc}_w(z, (\overline{p}, \psi q[i])).$$

As a consequence, $F(\overline{p}, \psi q[i])$ is a nonempty subset of $\{y, z\}$. □

Proposition 20. *Assume that $n \geqslant 3$ and $m \geqslant 4$, $m \neq 5$. If F is a unanimous positional* SCC, *then F is* US.

Proof. Let F be a unanimous positional SCC. Then, there exists a scoring vector w such that $F = PS_w$ and $w_1 > w_2$. We divide the proof into two different cases.

Assume first that m is even. Let $\overline{p} \in \mathcal{L}(A)^{I \setminus \{m\}}$ be defined as follows:

- for every $j \in [\frac{m}{2} - 1]$, $\overline{p}(j) = [1, 2, 3, (4), \ldots, (n)]$;
- for every $j \in \{\frac{m}{2}, \ldots, m - 2\}$, $\overline{p}(j) = [2, 1, 3, (4), \ldots, (n)]$;
- $\overline{p}(m - 1) = [2, 3, 1, (4), \ldots, n]$.

Consider then $q = [3, 1, 2, (4), \ldots, (n)]$ and, for every $x \in A$, set $S(x) \coloneqq \mathrm{sc}_w(x, (\overline{p}, q[m]))$. We have that

$$
\begin{aligned}
S(1) &= \frac{m-2}{2} w_1 + \frac{m-2}{2} w_2 + w_2 + w_3 \\
S(2) &= \frac{m-2}{2} w_1 + \frac{m-2}{2} w_2 + w_1 + w_3 \\
S(3) &= (m-2) w_3 + w_2 + w_1
\end{aligned}
$$

and, for every $u \in A \backslash \{1, 2, 3\}$, $S(u) \leqslant m w_3$. Recalling that $w_1 > w_2$ and $m \geqslant 4$, we deduce that $F(\overline{p}, q[m]) = \{2\}$. Note also that $\mathrm{rank}_q(3) + 1 = \mathrm{rank}_q(1) < \mathrm{rank}_q(2)$. A computation finally shows that $F(\overline{p}, \psi q[m]) = \{1, 2\}$, where ψ is the transposition that exchanges 1 and 3. Hence, F satisfies US.

Assume now that m is odd. Then we have $m \geqslant 7$. Let $\overline{p} \in \mathcal{L}(A)^{I \backslash \{m\}}$ be defined as follows:

- for every $j \in [\frac{m-3}{2}]$, $\overline{p}(j) = [1, 2, 3, (4), \ldots, (n)]$;
- for every $j \in \{\frac{m-3}{2} + 1, \ldots, m-3\}$, $\overline{p}(j) = [2, 1, 3, (4), \ldots, (n)]$;
- $\overline{p}(m-2) = [3, 2, 1, (4), \ldots, (n)]$;
- $\overline{p}(m-1) = [2, 1, 3, (4), \ldots, (n)]$.

Consider then $q = [3, 1, 2, (4), \ldots, (n)]$ and, for every $x \in A$, set $S(x) \coloneqq \mathrm{sc}_w(x, (\overline{p}, q[m]))$. We have that

$$
\begin{aligned}
S(1) &= \frac{m-3}{2} w_1 + \frac{m-3}{2} w_2 + w_3 + 2w_2 \\
S(2) &= \frac{m-3}{2} w_1 + \frac{m-3}{2} w_2 + w_1 + w_2 + w_3 \\
S(3) &= (m-2) w_3 + 2w_1
\end{aligned}
$$

and, for every $u \in A \backslash \{1, 2, 3\}$, $S(u) \leqslant m w_3$. Recalling that $w_1 > w_2$ and $m \geqslant 7$, we deduce $F(\overline{p}, q[m]) = \{2\}$. Note also that $\mathrm{rank}_q(3) + 1 = \mathrm{rank}_q(1) < \mathrm{rank}_q(2)$. A computation finally shows that $F(\overline{p}, \psi q[m]) = \{1, 2\}$, where ψ is the transposition that exchanges 1 and 3. Hence, F satisfies US. □

Corollary 21. *Assume that $n \geqslant 3$, $m \geqslant 4$ and $m \neq 5$. If F is a unanimous positional* SCC, *then F is Ω^F-**K**-manipulable.*

Proof. Apply Theorem 16 and Propositions 19 and 20. □

Proposition 22. *Assume that $n \geqslant 3$ and $m = 5$. If F is a unanimous positional* SCC, *then F is Ω^F-**K**-manipulable.*

Proof. Let F be a unanimous positional SCC. Then, there exists a scoring vector w such that $F = PS_w$ and $w_1 > w_2$.

Assume first that $w_2 > w_3$. In such a case, we can use the same argument of the second part of the proof of Proposition 20 to obtain that F satisfies US. As a consequence, by Theorem 16 and Propositions 19, we get that F is Ω^F-**K**-manipulable.

Assume now that $w_2 = w_3$. Consider $q = [3, 1, 2, (4), \ldots, (n)]$, $q' = [1, 3, 2, (4), \ldots, (n)]$ and $\omega = \{\overline{p} \in \mathcal{L}(A)^{I \backslash \{5\}} : F(\overline{p}, q[5]) = \{1, 2\}\}$. First, let us prove that $\omega \in \Omega_5^F(q)$. Indeed, let $\overline{p}' \in \mathcal{L}(A)^{I \backslash \{5\}}$ be such that $\overline{p}'(1) = [1, 2, 3, (4), \ldots, (n)]$, $\overline{p}'(2) = [2, 1, 3, (4), \ldots, (n)]$, $\overline{p}'(3) = [1, 2, 3, (4), \ldots, (n)]$, $\overline{p}'(4) = [2, 1, 3, (4), \ldots, (n)]$. Moreover, set, for every $x \in A$, $S'(x) \coloneqq \mathrm{sc}_w(x, (\overline{p}', q[5]))$. Recalling that $w_2 = w_3$, we have that $S'(1) = 2w_1 + 3w_2$, $S'(2) = 2w_1 + 3w_2$, $S'(3) = w_1 + 4w_2$, and, for every $u \in A \backslash \{1, 2, 3\}$, $S'(u) \leqslant 5w_2$. Recalling also that $w_1 > w_2$, we deduce $F(\overline{p}', q[5]) = \{1, 2\}$. Thus, $\overline{p}' \in \omega$, and so $\omega \neq \varnothing$ and $\omega \in \Omega_5^F(q)$.

Consider now any $\overline{p} \in \omega$. We have $\mathrm{sc}_w(1, (\overline{p}, q[5])) = \mathrm{sc}_w(2, (\overline{p}, q[5])) > \mathrm{sc}_w(u, (\overline{p}, q[5]))$ for all $u \neq A \backslash \{1, 2\}$. It easily follows that $F(\overline{p}, q'[5]) = \{1\} \succ_{\mathbf{K}(q)} \{1, 2\} = F(\overline{p}, q[5])$. Thus, we conclude that F is Ω^F-**K**-manipulable. □

Proof of Theorem 12. Apply Corollary 21 and Proposition 22. □

Proof of Theorem 9. If $n = 2$, then apply Theorem 13.

Assume now $n \geqslant 3$ and $m = 2$. Let $q = [2, 1, 3, (4), \dots, (n)]$, $q' = [2, 3, 1, (4), \dots, (n)]$, $\omega = \{\overline{p} \in \mathcal{L}(A)^{I\setminus\{2\}} : F(\overline{p}, q[2]) = \{1, 2\}\}$, and $\overline{p}' \in \mathcal{L}(A)^{I\setminus\{2\}}$ be such that $\overline{p}'(1) = [1, 2, 3, (4), \dots, (n)]$. We have $F(\overline{p}', q[2]) = \{1, 2\}$, hence $\omega \neq \varnothing$ and $\omega \in \Omega_2^F(q)$. We show that BO is Ω^{BO}-$\mathbf{K}$-manipulable proving that, for every $\overline{p} \in \omega$, $BO(\overline{p}, q'[2]) \succ_{\mathbf{K}(q)} BO(\overline{p}, q[2])$. Let $\overline{p} \in \omega$. Thus, $BO(\overline{p}, q[2]) = \{1, 2\}$, and then $\mathrm{bo}(1, (\overline{p}, q[2])) = \mathrm{bo}(2, (\overline{p}, q[2])) > \mathrm{bo}(x, (\overline{p}, q[2]))$ for all $x \in A\setminus\{1, 2\}$. Suppose that $\mathrm{bo}(1, (\overline{p}, q[2])) = \mathrm{bo}(3, (\overline{p}, q[2])) + 1$. We get

$$(n - \mathrm{rank}_{\overline{p}(1)}(1)) + (n - 2) = (n - \mathrm{rank}_{\overline{p}(1)}(3)) + (n - 3) + 1,$$

that is, $\mathrm{rank}_{\overline{p}(1)}(1) = \mathrm{rank}_{\overline{p}(1)}(3)$, a contradiction. Thus, we have $\mathrm{bo}(1, (\overline{p}, q[2])) \geqslant \mathrm{bo}(3, (\overline{p}, q[2])) + 2$. It is easily observed that $\mathrm{bo}(1, (\overline{p}, q'[2])) = \mathrm{bo}(1, (\overline{p}, q[2])) - 1$; $\mathrm{bo}(2, (\overline{p}, q'[2])) = \mathrm{bo}(2, (\overline{p}, q[2]))$; $\mathrm{bo}(3, (\overline{p}, q'[2])) = \mathrm{bo}(3, (\overline{p}, q[2])) + 1$; for all $x \in A\setminus\{1, 2, 3\}$, $\mathrm{bo}(x, (\overline{p}, q'[2])) = \mathrm{bo}(x, (\overline{p}, q[2]))$. That implies $BO(\overline{p}, q'[2]) = \{2\}$ and hence we conclude that $BO(\overline{p}, q'[2]) = \{2\} \succ_{\mathbf{K}(q)} \{1, 2\} = BO(\overline{p}, q[2])$.

Assume now $n \geqslant 3$ and $m = 3$. We prove that BO is US and we complete the proof applying Proposition 19 and Theorem 16. Let $\overline{p} \in \mathcal{L}(A)^{I\setminus\{3\}}$ be defined by $\overline{p}(1) = \overline{p}(2) = [3, 1, 2, (4), \dots, (n)]$ and $q = [2, 1, 3, (4), \dots, (n)] \in \mathcal{L}(A)$. Observe that $BO(\overline{p}, q[3]) = \{3\}$, $\mathrm{rank}_q(2) + 1 = \mathrm{rank}_q(1) < \mathrm{rank}_q(3)$, and $BO(\overline{p}, \psi q\,[3]) = \{1, 3\}$, where ψ is the transposition that exchanges 1 and 2. Thus, BO is US.

Finally, if $n \geqslant 3$ and $m \geqslant 4$, then apply Theorem 12. □

Proposition 23. *Assume that $n \geqslant 3$ and $m \in \{2, 3\}$. Then PL is $\mathbf{K}$-strategy-proof.*

Proof. For every $q \in \mathcal{L}(A)$, we denote by $\mathrm{top}(q)$ the best alternative in q.

Assume first that $I = \{1, 2\}$. Let $i \in I$, $\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$, and $q \in \mathcal{L}(A)$. Denote by j the unique element in $I\setminus\{i\}$. Of course, we have $\mathrm{top}(q) \in PL(\overline{p}, q[i])$. We split the proof in two cases.

- If $\mathrm{top}(p(j)) = \mathrm{top}(q)$, then $PL(\overline{p}, q[i]) = \{\mathrm{top}(q)\}$. In that case, for every $q' \in \mathcal{L}(A)$, we have $PL(\overline{p}, q'[i]) \not\succ_{\mathbf{K}(q)} PL(\overline{p}, q[i])$.
- If $\mathrm{top}(p(j)) \neq \mathrm{top}(q)$, then we have $PL(p) = \{\mathrm{top}(p(j)), \mathrm{top}(q)\}$. Consider $q' \in \mathcal{L}(A)$. If $\mathrm{top}(q') = \mathrm{top}(q)$, we have $PL(\overline{p}, q'[i]) = PL(\overline{p}, q[i])$. If $\mathrm{top}(q') \neq \mathrm{top}(q)$, we have $PL(\overline{p}, q'[i]) = \{\mathrm{top}(p(j)), \mathrm{top}(q')\}$. In both cases, we have $PL(\overline{p}, q'[i]) \not\succ_{\mathbf{K}(q)} PL(\overline{p}, q[i])$.

Assume then that $I = \{1, 2, 3\}$. We first observe that, for every $p \in \mathcal{L}(A)^I$, we have that $|PL(p)| = 1$ or $|PL(p)| = 3$. Indeed, assume first that, for every distinct $i, j \in I$, we have $\mathrm{top}(p(i)) \neq \mathrm{top}(p(j))$. Then $|PL(p)| = 3$. Assume next that there exist $x \in A$ and $i, j \in I$ distinct such that $x = \mathrm{top}(p(i)) = \mathrm{top}(p(j))$. Then $PL(p) = \{x\}$. Consider now $i \in I$, $\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$, and $q \in \mathcal{L}(A)$.

- If $|PL(\overline{p}, q[i])| = 3$, then we have $\mathrm{top}(q) \in PL(\overline{p}, q[i])$. Let $q' \in \mathcal{L}(A)$. If $\mathrm{top}(q') = \mathrm{top}(q)$, then $PL(\overline{p}, q'[i]) = PL(\overline{p}, q[i])$ and that implies $PL(\overline{p}, q'[i]) \not\succ_{\mathbf{K}(q)} PL(\overline{p}, q[i])$. If instead $\mathrm{top}(q') \neq \mathrm{top}(q)$, then we have $\mathrm{top}(q) \notin PL(\overline{p}, q'[i])$. Since $\mathrm{top}(q) \in PL(\overline{p}, q[i])$, we have $PL(\overline{p}, q'[i]) \not\succ_{\mathbf{K}(q)} PL(\overline{p}, q[i])$.
- If $|PL(\overline{p}, q[i])| = 1$, then we have two possibilities: if $PL(\overline{p}, q[i]) = \{\mathrm{top}(q)\}$, then, for every $C \in \mathcal{P}_0(A)$, we have that $C \not\succ_{\mathbf{K}(q)} \{\mathrm{top}(q)\}$. We deduce that, for every $q' \in \mathcal{L}(A)$, we have $PL(\overline{p}, q'[i]) \not\succ_{\mathbf{K}(q)} PL(\overline{p}, q[i])$. If $PL(\overline{p}, q[i]) = \{x\}$, with $x \neq \mathrm{top}(q)$ then, for both the individuals in $I\setminus\{i\}$, x is the best alternative. As a consequence, for any q', we have $\{x\} = PL(\overline{p}, q'[i])$ and thus $PL(\overline{p}, q'[i]) \not\succ_{\mathbf{K}(q)} PL(\overline{p}, q[i])$.

That proves that PL is $\mathbf{K}$-strategy-proof. □

Proof of Theorem 10. Apply Theorem 12 and Proposition 23. □

C Proof of Theorem 11

Lemma 24. *The following facts hold:*

(*i*) *For every* $p \in \mathcal{L}(A)^I$*, we have* $|NP(p)| \geqslant \max\{1, n-m\}$*;*

(*ii*) $|NP(p)| \geqslant 2$ *for all* $p \in \mathcal{L}(A)^I$ *if and only if* $n - m \geqslant 2$*.*

Proof. For every $p \in \mathcal{L}(A)^I$, define $Z(p) := \{z \in A : \exists i \in I \text{ such that } \mathrm{rank}_{p(i)}(z) = n\}$.

(*i*) Clearly we have $|Z(p)| \leqslant \min\{n, m\}$. Now, observe that $A \backslash Z(p) \subseteq NP(p)$ and thus

$$|NP(p)| \geqslant |A| - |Z(p)| \geqslant n - \min\{n, m\} = \begin{cases} 0 & \text{if } n \leqslant m \\ n-m & \text{if } n > m \end{cases} = \max\{0, n-m\}$$

Since $NP(p) \neq \varnothing$, we deduce $|NP(p)| \geqslant \max\{1, n-m\}$.

(*ii*) If $n - m \geqslant 2$, by (*i*), we have $|NP(p)| \geqslant n - m \geqslant 2$ for all $p \in \mathcal{L}(A)^I$. Assume next that $n - m \leqslant 1$, that is, $m \geqslant n-1$. Then, there exists $p \in \mathcal{L}(A)^I$ such that, for every $i \in I$, $\mathrm{rank}_{p(i)}(n) = 1$ and $Z(p) = [\![n-1]\!]$. Of course, $\mathrm{np}(n, p) = m$ and $\mathrm{np}(x, p) \leqslant m - 1$ for all $x \in [\![n-1]\!]$. As a consequence, $NP(p) = \{n\}$. □

Proposition 25. *Assume that* $n - m \geqslant 2$*. Then* NP *is* **K**-*strategy-proof.*

Proof. Let $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\overline{p} \in \mathcal{L}(A)^{I \backslash \{i\}}$. We set $p := (\overline{p}, q\,[i])$, $p' := (\overline{p}, q'\,[i])$, $B := NP(p)$ and $B' := NP(p')$. Moreover, for every $y \in A$, we set $N(y) := \mathrm{np}(y, p)$ and $N'(y) := \mathrm{np}(y, p')$.

In order to show that NP is **K**-strategy-proof, we need to show that

$$B' \not\succ_{\mathbf{K}(q)} B. \tag{3}$$

If $|B \cap B'| \geqslant 2$, then, by Proposition 18(*i*), we immediately get (3). If instead $|B \cap B'| \in \{0, 1\}$, then, by Lemma 24(*ii*), we know that $|B| \geqslant 2$ and $|B'| \geqslant 2$. As a consequence, $B \nsubseteq B'$ and $B' \nsubseteq B$. We divide the argument into the two cases $|B \cap B'| = 0$ and $|B \cap B'| = 1$.

Assume first that $|B \cap B'| = 0$. Suppose that there exists $x' \in B'$ such that $\mathrm{rank}_q(x') = n$. Pick $x \in B$. Since $B \cap B' = \varnothing$, we deduce $x \succ_q x'$. Thus, by Proposition 18(*iii*), we deduce (3). Suppose instead that, for every $x' \in B'$, $\mathrm{rank}_q(x') \leqslant n - 1$. Let $x' \in B'$ and $x \in B$. Since $B \cap B' = \varnothing$, we have that $x' \notin B$ and then $N(x') < N(x)$. Thus, we deduce $N'(x') \leqslant N(x') < N(x)$, and then $N'(x') \leqslant N(x) - 1 \leqslant N'(x)$. Since $x' \in B'$, we conclude that $x \in B'$, a contradiction.

Assume now that $|B \cap B'| = 1$ and let $B \cap B' = \{x\}$. Since B, B' are not included one in the other and have size at least 2, there exist $z \in B \backslash \{x\}$ and $z' \in B' \backslash \{x\}$. Then $z' \notin B$ and we have

$$N(x) = N(z) > N(z'). \tag{4}$$

Moreover, $z \notin B'$ and we have

$$N'(x) = N'(z') > N'(z). \tag{5}$$

Assume first that $\mathrm{rank}_q(x) = n$ and $\mathrm{rank}_{q'}(x) \leqslant n - 1$. Then we have $N'(x) > N(x)$ and thus $N(z') \geqslant N'(z') = N'(x) > N(x)$, against the fact that $x \in B$. Assume next that $\mathrm{rank}_q(x) \leqslant n - 1$ and $\mathrm{rank}_{q'}(x) = n$. Then, we have $N(x) > N'(x)$ and thus $N'(z) \geqslant N(z) = N(x) > N'(x)$, against the fact $x \in B'$. Assume now that we have both $\mathrm{rank}_q(x) \leqslant n - 1$ and $\mathrm{rank}_{q'}(x) \leqslant n - 1$, or both $\mathrm{rank}_q(x) = n$ and $\mathrm{rank}_{q'}(x) = n$. Then, we have $N(x) = N'(x)$. As a consequence, using (5), we get $N(z) = N(x) = N'(x) > N'(z)$, and so we deduce that z is not the worst alternative for q; using also (4), we get $N'(z') = N'(x) = N(x) > N(z')$, and so we deduce that z' is the worst alternative for q. We then conclude that $z \succ_q z'$ and, by Proposition 18(*iii*), we finally obtain (3). □

Proposition 26. *Assume that* $m \geqslant n - 1$*. Then* NP *is* **K**-*manipulable.*

Proof. For every $p \in \mathcal{L}(A)^I$, we set $Z(p) := \{z \in A : \exists i \in I \text{ such that } \mathrm{rank}_{p(i)}(z) = n\}$. In order to prove that NP is $\mathbf{K}$-manipulable, we exhibit $p \in \mathcal{L}(A)^I$, $i \in I$ and $q' \in \mathcal{L}(A)$ such that

$$NP(p_{|_{I\setminus\{i\}}}, q'[i]) \succ_{\mathbf{K}(p(i))} NP(p). \tag{6}$$

Since $n \geqslant 3$ and $m \geqslant n-1$, we have $A\setminus\{1,2\} \neq \varnothing$ and $m \geqslant n-2$. Thus, there exists $p \in \mathcal{L}(A)^I$ such that $Z(p) = A\setminus\{1,2\}$ and $1 \succ_{p(i)} 2$ for all $i \in I$. Of course, $NP(p) = \{1,2\}$. Let $z^* \in Z(p)$ be an alternative that is the worst alternative for the maximum number of individuals according to p. Since $m > n-2$, we have that z^* is the worst alternative for at least two individuals.

Let i be one of the individuals that considers z^* her worst alternative and let ψ be the transposition that exchanges 2 and z^*. Define $q' := \psi p(i) \in \mathcal{L}(A)$ and $p' := (p_{|_{I\setminus\{i\}}}, q'[i])$. We have that $NP(p') = \{1\}$ since $\mathrm{np}(1,p') = m$, $\mathrm{np}(2,p') = m-1$, $\mathrm{np}(z^*,p') = \mathrm{np}(z^*,p) + 1 \leqslant (m-2)+1 = m-1$, and $\mathrm{np}(x,p') = \mathrm{np}(x,p) \leqslant m-1$ for all $x \in A\setminus\{1,2,z^*\}$. By Proposition 18($iv$), $1 \succ_{p(i)} 2$ implies $\{1\} \succ_{\mathbf{K}(p(i))} \{1,2\}$, and hence (6) is finally shown. ☐

Lemma 27. *Let $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\omega \in \Omega_i^{NP}(q)$. Let $B \in P_0(A)$ be the set such that*

$$\omega = \{\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}} : NP(\overline{p}, q[i]) = B\}.$$

Let z be the worst alternative in q and z' be the worst alternative in q'. If one of the following conditions is satisfied

- $z = z'$,
- $z \in B$,
- $|B| \leqslant n-2$,

then one of the following facts hold:

(i) *for every $\overline{p} \in \omega$, $NP(\overline{p}, q'[i]) \not\succ_{\mathbf{K}(q)} B$;*

(ii) *there exists $\overline{p}' \in \omega$ such that $NP(\overline{p}', q'[i]) \not\succsim_{\mathbf{K}(q)} B$.*

Proof. Assume first that $z = z'$. We show that (i) holds. Let $\overline{p} \in \omega$. For every $x \in A$, we have $\mathrm{np}(x, (\overline{p}, q[i])) = \mathrm{np}(x, (\overline{p}, q'[i]))$, thus $NP(\overline{p}, q'[i]) = B \not\succ_{\mathbf{K}(q)} B$.

Assume next that $z \neq z'$ and $z \in B$. We show that (i) holds. Let $\overline{p} \in \omega$. Define $p := (\overline{p}, q[i])$ and $p' := (\overline{p}, q'[i])$. Since $z \in B = NP(p)$ we have that, for every $x \in A\setminus\{z\}$,

$$\mathrm{np}(z,p) \geqslant \mathrm{np}(x,p). \tag{7}$$

Moreover, since z is not the worst alternative in q', we have

$$\mathrm{np}(z,p') = \mathrm{np}(z,p) + 1 \tag{8}$$

and, for every $x \in A\setminus\{z\}$,

$$\mathrm{np}(x,p') \leqslant \mathrm{np}(x,p). \tag{9}$$

As a consequence, by (8), (7), (9), we get $\mathrm{np}(z,p') > \mathrm{np}(z,p) \geqslant \mathrm{np}(x,p) \geqslant \mathrm{np}(x,p')$ for all $x \in A\setminus\{z\}$, which gives $NP(p') = \{z\} \not\succ_{\mathbf{K}(q)} B$.

Assume finally that $z \neq z'$, $z \notin B$ and $|B| \leqslant n-2$. We show that (ii) holds. Consider $C := A\setminus B$. Thus, $z \in C$ and $|C| \geqslant 2$. For every $\overline{p} \in \mathcal{L}(A)^{I\setminus\{i\}}$, define

$$Z(\overline{p}) := \{w \in A : \exists j \in I\setminus\{i\} \text{ such that } \mathrm{rank}_{p(j)}(w) = n\}.$$

By Lemma 24 (i), we have $|B| \geqslant n-m$, hence

$$|C\setminus\{z\}| = |C| - 1 = n - |B| - 1 \leqslant n + m - n - 1 = m - 1.$$

Thus, we can construct $\overline{p}' \in \mathcal{L}(A)^{I\setminus\{i\}}$ such that $Z(\overline{p}') = C\setminus\{z\}$. Clearly we have $NP(\overline{p}', q[i]) = B$ and hence $\overline{p}' \in \omega$. Set now $p := (\overline{p}', q[i])$ and $p' := (\overline{p}', q'[i])$. Recalling that z is not the worst alternative in q', we have $\mathrm{np}(z,p') = \mathrm{np}(z,p) + 1 = (m-1)+1 = m$. As a consequence, $z \in NP(p')$. Since $z \notin B$ we have $NP(\overline{p}', q'[i]) \not\succsim_{\mathbf{K}(q)} B$. ☐

Proposition 28. *Assume that n divides $m-1$ or $n \geqslant 4$. Then NP is Ω^{NP}-$\mathbf{K}$-strategy-proof.*

Proof. Let us consider $i \in I$, $q, q' \in \mathcal{L}(A)$ and $\omega \in \Omega_i^{NP}(q)$ and prove that one of the following facts hold:

(a) for every $\overline{p} \in \omega$, $NP(\overline{p}, q'[i]) \nsucc_{\mathbf{K}(q)} B$,

(b) there exists $\overline{p}' \in \omega$ such that $NP(\overline{p}', q'[i]) \nsucceq_{\mathbf{K}(q)} B$,

where $B \in P_0(A)$ is such that $\omega = \{\overline{p} \in \mathcal{L}(A)^{I\backslash\{i\}} : NP(\overline{p}, q[i]) = B\}$. Let z be the worst alternative in q, and z' be the worst alternative in q'.

If $z = z'$ or $z \in B$ or $|B| \leqslant n-2$, then, by Lemma 27, we know that one between (a) and (b) holds.

Assume next that $z \neq z'$, $z \notin B$ and $|B| \geqslant n-1$. Since $z \notin B$, we actually have $|B| = n-1$. Note also that, under these assumptions, $B = A\backslash\{z\}$.

- Assume that n divides $m-1$. We prove that (b) holds. Defining $c \coloneqq \frac{m-1}{n}$, we can consider $\overline{p}' \in \mathcal{L}(A)^{I\backslash\{i\}}$ such that, for every $x \in A$, x is the worst alternative of exactly c individuals. Since z is the worst alternative in q, we have $NP(\overline{p}', q[i]) = B$ and then $\overline{p}' \in \omega$. Since $z \neq z'$, we have $NP(\overline{p}', q'[i]) = A\backslash\{z'\}$. Since $A\backslash\{z'\} \nsucceq_{\mathbf{K}(q)} B$, we conclude that $NP(\overline{p}', q'[i]) \nsucceq_{\mathbf{K}(q)} B$.

- Assume that $n \geqslant 4$. We prove that (b) holds. Consider $\overline{p}' \in \mathcal{L}(A)^{I\backslash\{i\}}$ such that z is the worst alternative of all individuals. Of course, $NP(\overline{p}', q[i]) = B$ and then $\overline{p}' \in \omega$. Since $z' \neq z$, we have that $NP(\overline{p}', q'[i]) = A\backslash\{z, z'\} \subsetneq B$. Note that $|A\backslash\{z, z'\}| = n-2 \geqslant 2$. By Proposition 18($ii$), we deduce that B and $NP(p')$ are not $\mathbf{K}(q)$-comparable. It follows that $NP(\overline{p}', q'[i]) \nsucceq_{\mathbf{K}(q)} B$.

□

Proposition 29. *Assume that $n = 3$ and 3 does not divide $m-1$. Then NP is Ω^{NP}-$\mathbf{K}$-manipulable;*

Proof. Let $i \in I$, $q \coloneqq [1, 2, 3]$ and $\omega \coloneqq \{\overline{p} \in \mathcal{L}(A)^{I\backslash\{i\}} : NP(\overline{p}, q[i]) = \{1, 2\}\}$. Let $\overline{p}' \in \mathcal{L}(A)^{I\backslash\{i\}}$ be such that 3 is the worst alternative of all individuals. Thus we have $\overline{p}' \in \omega$, and hence $\omega \neq \varnothing$ and $\omega \in \Omega_i^{NP}(q)$.

Consider now $\overline{p} \in \omega$. Set $p \coloneqq (\overline{p}, q[i])$ and, for every $x \in A$, $N(x) \coloneqq \mathrm{np}(x, p)$ and

$$L(x) \coloneqq |\{i \in I : \mathrm{rank}_{p(i)}(x) = 3\}|.$$

Note that $L(x) = m - N(x)$ for all $x \in A$, $\sum_{x \in A} L(x) = m$, and $L(1) = L(2) < L(3)$. Assume by contradiction $L(3) = L(1) + 1$. Then, we get

$$m = \sum_{x \in A} L(x) = L(1) + L(2) + L(3) = 2L(1) + L(1) + 1 = 3L(1) + 1.$$

Thus $n = 3$ divides $m - 1$, a contradiction. Hence, $L(3) \geqslant L(1) + 2$ and $L(3) \geqslant L(2) + 2$, and so $N(3) \leqslant N(1) - 2$ and $N(3) \leqslant N(1) - 2$. Let ψ be the transposition that exchanges 3 and 2 and set, for every $x \in A$, $N'(x) \coloneqq \mathrm{np}(x, (\overline{p}, \psi q[i]))$. We have $N'(3) = N(3) + 1$, $N'(2) = N(2) - 1$ and, $N'(1) = N(1)$. Then $NP(\overline{p}, \psi q[i]) = \{1\} \succ_{\mathbf{K}(q)} \{1, 2\}$. Thus, NP is Ω^{NP}-$\mathbf{K}$-manipulable. □

Proof of Theorem 11. Apply Propositions 25, 26, 28 and 29. □

References

Barberà, S., 1977a. Manipulation of social decision functions. Journal of Economic Theory 15, 266-278.

Barberà, S., 1977b. The manipulation of social choice mechanisms that do not leave "too much" to chance. Econometrica 45, 1573-1588.

Barberà, S., Dutta, B., Sen, A., 2001. Strategy-proof social choice correspondences. Journal of Economic Theory 101, 374-394.

Barberà, S., Bossert, W., Pattanaik, P.K., 2004. Ranking sets of objects. In: Barberà, S., Hammond, P.J., Seidl, C. (eds) Handbook of Utility Theory. Springer, Boston, MA.

Barberà, S., 2011. Strategy-proof social choice. Handbook of Social Choice and Welfare, volume 2, 731-831.

Brandt, F., Brill, M., 2011. Necessary and sufficient conditions for the strategy-proofness of irresolute social choice functions. In Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge (TARK XIII). Association for Computing Machinery, New York, NY, USA, 136-142.

Campbell, D.E., Kelly, J.S., Qi, S., 2018. A stability property in social choice theory. International Journal of Economic Theory 14, 85-95.

Ching, S., Zhou,, L., 2002. Multi-valued strategy-proof social choice rules. Social Choice and Welfare 19, 569-580.

Conitzer, V., Walsh, T., Xia, L., 2011. Dominating manipulations in voting with partial information. In: Proceedings of the Twenty-Fifth AAAI Conference on Artificial Intelligence, 638-643.

Duggan, J., Schwartz, T., 2000. Strategic manipulability without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized. Social Choice and Welfare 17, 85-93.

Endriss, U., Obraztsova, S., Polukarov, M., Rosenschein, J.S., 2016. Strategic voting with incomplete information. In: Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence. IJCAI-16, pp. 236-242.

Gärdenfors, P., 1976. Manipulation of social choice functions. Journal of Economic Theory 13, 217-228.

Gibbard, A., 1973. Manipulation of voting schemes: A general result. Econometrica 41, 587-601.

Gori, M., 2021. Manipulation of social choice functions under incomplete information. Games and Economic Behavior 129, 350-369.

Kelly, J.S., 1977. Strategy-proofness and social choice functions without singlevaluedness. Econometrica 45, 439-446.

Nurmi, H., 1987. Comparing Voting Systems. D. Reidel Publishing Company, Dordrecht.

Pattanaik, P.K., 1975. Strategic voting without collusion under binary and democratic group decision rules. The Review of Economic Studies 42, 93-103.

Reijngoud, A., Endriss, U., 2012. Voter response to iterated poll information. In: Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems. AAMAS, pp. 635-644.

Sato, S., 2008. On strategy-proof social choice correspondences. Social Choice and Welfare 31, 331-343.

Satterthwaite, M.A., 1975. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. Journal of Economic Theory 10, 187-217.

Taylor, A.D., 2002. The manipulability of voting systems. The American Mathematical Monthly 109, 321-337.

Taylor, A.D., 2005. Social choice and the mathematics of manipulation. Cambridge: Cambridge University Press.

Terzopoulou, Z., Endriss, U., 2019. Strategyproof judgment aggregation under partial information. Social Choice and Welfare 53, 415-442.

Tsiaxiras, J., 2021. Strategic voting under incomplete information in approval-based committee elections. MSc Thesis. University of Amsterdam. Available at https://eprints.illc.uva.nl/id/eprint/1799/1/MoL-2021-09.text.pdf

Veselova, Y.A., 2020. Does incomplete information reduce manipulability?. Group Decision and Negotiation 29, 523-548.

Veselova, Y.A., Karabekyan, D., 2023. Simultaneous manipulation under incomplete information. Working paper. Available at https://api.semanticscholar.org/CorpusID:262231173